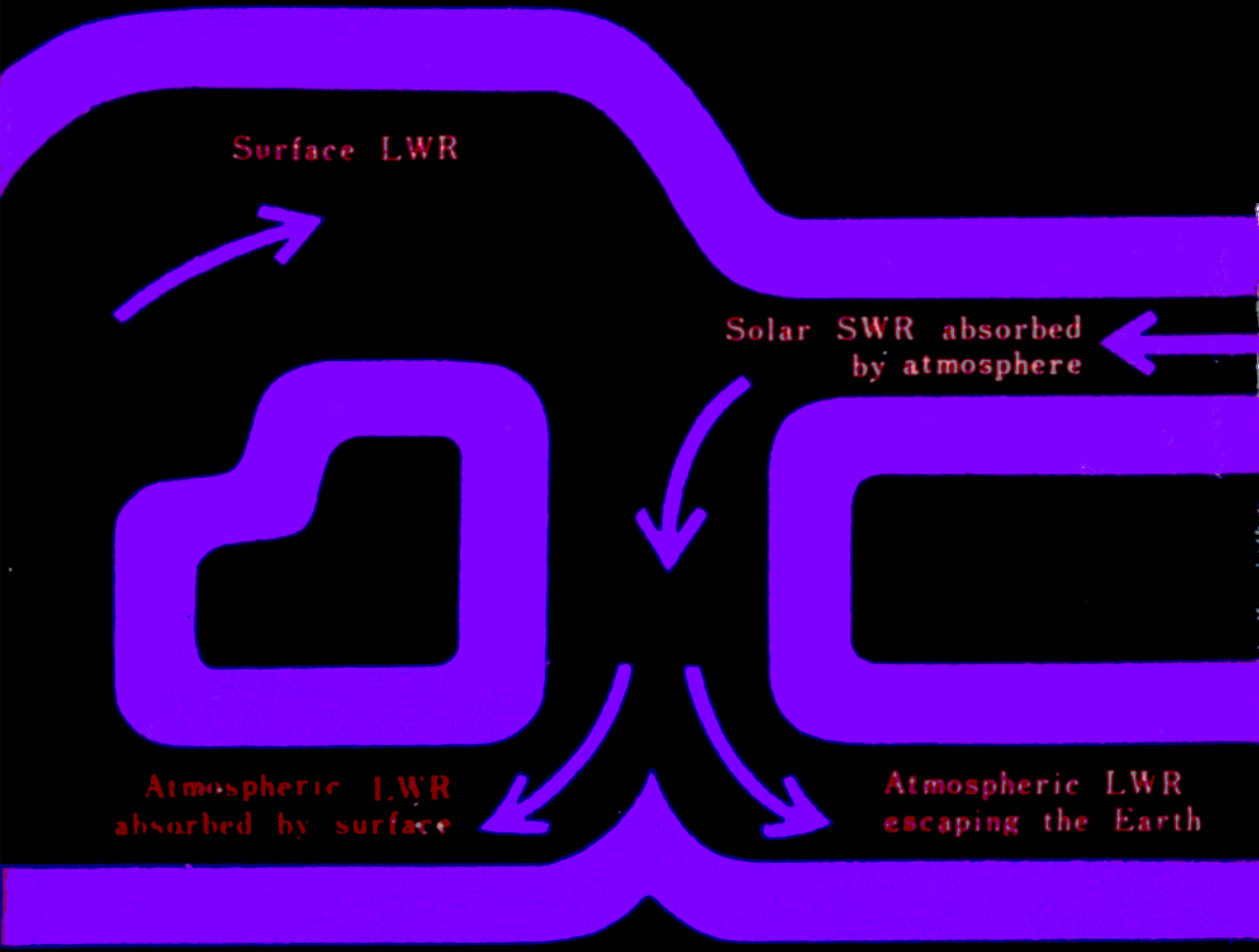
B.M.Smirnov

Physics of Weakly Ionized Gases


Surface LWR
Solar SWR absorbed by atmosphere
Atmospheric LWR absorbed by surface
Atmospheric LWR escaping the Earth

Mir Publishers Moscow

ABOUT THE BOOK

Each section of the book contains the theoretical part and problems with solutions. More than 200 problems on generation and motion of charged particles in plasma, plasma oscillations and the magnetic and electric properties of plasma are discussed in the book. The author aims at revealing the physical meaning of the problems and, at the same time, acquainting the reader with the procedures and methods employed for solving such problems. The book is intended for the university students specializing in physics and related fields. It can be used also as a handbook by the practical workers in these fields, since many of the problems are based on recently published results and the appendices contain the necessary reference data.

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Doctor of physico-mathematical sciences Boris Mikhailovich Smirnov has been teaching physics at the Moscow Power-Engineering Institute for ten years. His fields of research are the physics of weakly ionized gases and the elementary processes in plasmas. He has more than one hundred publications to his credit, among them textbooks, aids for young specialists, and about ten monographs.

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→ Solar SWR absorbed by surface

·ИЗДАТЕЛЬСТВО·МИР·

Б. М. Смирнов

Физика слабоионизованного газа

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(problems and solutions)

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PREFACE

This is the translation of the second Russian edition of the book. The experience with the first edition has shown that presentation of the material in the form of problems (there are more than 200 problems in the book) is particularly suitable for independent study of the course of low-temperature plasma physics or some questions of it, so that the book serves as a useful addition to the available textbooks on physics of weakly ionized plasma. The problems in the book either deal with separate theoretical subjects or analyze individual research questions. The emphasis is on presenting the totality of methods and approaches employed for solving such problems without mathematics obscuring the physical nature of the processes involved. At the same time, the solutions of the problems present scientific information of interest for the workers in this field (many problems stem from recent research results and are formulated so as to be of possible use for practical work). The appendices containing reference data serve the same purpose.

Thus, the book is aimed at two types of readers. The first type is graduate students who can study the problems in the book to get acquainted with the field and its theory. The second type is physicists or technologists working with weakly ionized plasmas, who can make use of the information contained in the book.

The bibliography given at the end of the book is a list of textbooks, monographs and reviews where the reader can find more detailed information on the subjects covered by the book. Some of the problems are based on original research results but no references are given in such cases because such problems are formulated primarily for the teaching purposes and no detailed discussion of the results is given.

The appendices serve different purposes. On the one hand, they briefly present the definitions and main relationships for collision cross sections and spectral line broadening. This information is used in the book. On the other hand, the appendices contain some reference data including information on physical constants, simple physical relationships, and selected parameters of atoms, molecules and ions.

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CHAPTER 1

THE BOLTZMANN KINETIC EQUATION

Let us consider an ideal gas. The state of a particle of this gas at a given moment of time is described by its velocity $\mathbf{v}$, coordinate $\mathbf{r}$, and the internal degrees of freedom which we shall denote by the generalized parameter I. The ideality of the gas implies that the mean distance $N^{-1/3}$ between the particles (here N is the density of the particles) is much larger than their interaction radius (the characteristic distance between the particles, at which the potential energy of interaction between the particles is of the order of their kinetic energy). This condition of gas ideality can be written as

$$N\sigma^{3/2} \ll 1 \tag{1.1}$$

where σ is the scattering cross section for the particles, at which their states are noticeably changed.

If the ideality condition (1.1) is satisfied, each particle travels as a free particle during most of the time and only for a small fraction ($\sim N\sigma^{3/2}$) of the time interacts with other particles. The state of the colliding particle changes during this short collision time. A simultaneous collision of three particles (that is, when they simultaneously approach each other to a distance of the order of the interaction radius) is a more rare event than collision of two particles. Therefore, when analyzing the motion of a particle in the ideal gas, we can limit ourselves to considering binary collisions of this particle with other gas particles.

The state of the gas can be conveniently described by the function $f(\mathbf{v}_1, \mathbf{r}, I_1, t)$ of distribution of the particles over the states; the number of particles in the unit volume around point $\mathbf{r}$, having the velocities from $\mathbf{v}_1$ to $\mathbf{v}_1 + d\mathbf{v}_1$ and the internal state described by the quantum number I_1, is $f(\mathbf{v}_1, \mathbf{r}, t, I_1)\, d\mathbf{v}_1$.

The distribution function for the particles is a solution of the equation known as the *Boltzmann kinetic equation*. Let us trace the particles which are at the point $\mathbf{r}$, travel with the velocity $\mathbf{v}_1$ and are in the internal state described by the quantum number I_1. The number of particles in a given state with a given velocity in the unit volume around a given point per unit time is known as the *collision integral* I_c. Then the variation of the distribution function during time dt is

$$df = f(\mathbf{v}_1 + d\mathbf{v}_1,\ \mathbf{r} + d\mathbf{r},\ t + dt,\ I_1) - f(\mathbf{v}_1, \mathbf{r}, t, I_1) = I_c\, dt$$

Since the variations of the coordinate and the velocity of the test particle are

$$d\mathbf{r} = \mathbf{v}_1\, dt \quad \text{and} \quad d\mathbf{v}_1 = \frac{\mathbf{F}}{m}\, dt$$

where $\mathbf{F}$ is the external force acting on the particle and m is the particle's mass, the above equation can be rewritten as

$$\frac{\partial f}{\partial t} + \mathbf{v}_1\, \nabla f + \frac{\mathbf{F}}{m}\frac{\partial f}{\partial \mathbf{v}_1} = I_c \tag{1.2}$$

§ 1.1. PROPERTIES OF THE KINETIC EQUATION

Problem 1.1. Express the collision integral in terms of the differential cross section of collisions between particles.

Let us define the constant $W(\mathbf{v}_1, I_1;\ \mathbf{v}_2, I_2 \rightarrow \mathbf{v}_1', I_1';\ \mathbf{v}_2', I_2')$ for collisions between particles in the states $\mathbf{v}_1, I_1$ and $\mathbf{v}_2, I_2$ in the course of which they go over to the states $\mathbf{v}_1', I_1'$ and $\mathbf{v}_2', I_2'$, respectively. The *principle of detailed balancing* (see A2.6) yields

$$W(\mathbf{v}_1, I_1;\ \mathbf{v}_2, I_2 \rightarrow \mathbf{v}_1', I_1';\ \mathbf{v}_2', I_2') = W(\mathbf{v}_1', I_1';\ \mathbf{v}_2', I_2' \rightarrow \mathbf{v}_1, I_1;\ \mathbf{v}_2, I_2) \tag{1.3}$$

If the gas consists of one species, the collision integral, by definition, is

$$I_c = \int (f_1' f_2' - f_1 f_2) W\, d\mathbf{v}_2\, d\mathbf{v}_1'\, d\mathbf{v}_2' \tag{1.4}$$

where $f_1 \equiv f(\mathbf{v}_1, \mathbf{r}, I_1, t)$ and the distribution functions f_2, f_1' and f_2' are defined in a similar way. Equation (1.4) implies summation over the internal quantum numbers I_2, I_1' and I_2'.

The number of integrations in eq. (1.4) can be reduced if the laws of conservation of momentum and energy are taken into account, which use the properties of the transition probability W. This can be done by considering the differential scattering cross section defined as the ratio of the number of particles scattered into the solid angle element per unit time per one scattering particle to the flux density of the scattered particles. The number of scattering events per unit time in the unit volume, in which the first particle changes its velocity from the range $\mathbf{v}_1, \mathbf{v}_1 + d\mathbf{v}_1$ to $\mathbf{v}_1', \mathbf{v}_1' + d\mathbf{v}_1'$ and the second particle, from the range $\mathbf{v}_2, \mathbf{v}_2 + d\mathbf{v}_2$ to $\mathbf{v}_2', \mathbf{v}_2' + d\mathbf{v}_2'$ with a given variation of the internal quantum numbers, is $f_1 f_2 W\, d\mathbf{v}_1\, d\mathbf{v}_2\, d\mathbf{v}_1'\, d\mathbf{v}_2'$.

The number of the scattering particles in the unit volume is $f_1 d\mathbf{v}_1$ and the flux density of the scattered particles is $|\mathbf{v}_2 - \mathbf{v}_1| f_2 d\mathbf{v}_2$, so that the differential scattering cross section is

$$d\sigma = \frac{f_1 f_2 W \, d\mathbf{v}_1 \, d\mathbf{v}_2 \, d\mathbf{v}_1' \, d\mathbf{v}_2'}{f_1 \, d\mathbf{v}_1 f_2 |\mathbf{v}_2 - \mathbf{v}_1| \, d\mathbf{v}_2} = \frac{W \, d\mathbf{v}_1' \, d\mathbf{v}_2'}{|\mathbf{v}_2 - \mathbf{v}_1|}$$

Substitution of this equation into eq. (1.4) yields the following modification of the Boltzmann kinetic equation (1.2):

$$\frac{\partial f_1}{\partial t} + \mathbf{v}_1 \frac{\partial f_1}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m} \frac{\partial f_1}{\partial \mathbf{v}_1} = \int (f_1' f_2' - f_1 f_2) \, |\mathbf{v}_1 - \mathbf{v}_2| \, d\sigma \, d\mathbf{v}_2 \tag{1.5}$$

As before, summation over the internal degrees of freedom I_1', I_2' and I_2 is implied. Equation (1.5) describes a one-component system. The collision integral for a multicomponent system must take into account the collisions of the test particle with the particles of each species.

Problem 1.2. Demonstrate that the functional $H(t) = \int f(\mathbf{v}_1, t) \ln f(\mathbf{v}_1, t) \, d\mathbf{v}_1$, defined for a gas on which no external fields are acting, satisfies the condition $dH/dt \leqslant 0$.

Differentiating $H(t)$ with respect to time and using the Boltzmann equation with the collision integral in the form (1.4), we obtain

$$\frac{dH}{dt} = \int d\mathbf{v}_1 \, d\mathbf{v}_2 \, d\mathbf{v}_1' \, d\mathbf{v}_2' W (f_1' f_2' - f_1 f_2)(1 + \ln f_1)$$

The integrand can be symmetrized by making the substitutions $\mathbf{v}_1 \to \mathbf{v}_2$, $\mathbf{v}_2 \to \mathbf{v}_1$:

$$\frac{dH}{dt} = \int d\mathbf{v}_1 \, d\mathbf{v}_2 \, d\mathbf{v}_1' \, d\mathbf{v}_2' W (f_1' f_2' - f_1 f_2) \left(1 + \frac{1}{2} \ln f_1 f_2\right)$$

Making the substitution $\mathbf{v}_{1,2} \to \mathbf{v}_{1,2}'$, $\mathbf{v}_{1,2}' \to \mathbf{v}_{1,2}$ in the integrand and using the principle of detailed balancing (1.3), we obtain

$$\frac{dH}{dt} = \int d\mathbf{v}_1 \, d\mathbf{v}_2 \, d\mathbf{v}_1' \, d\mathbf{v}_2' W (f_1' f_2' - f_1 f_2) \ln \frac{f_1 f_2}{f_1' f_2'}$$

The function of the type $(x - y) \ln (y/x)$ is non-positive for any positive x and y. Hence, the functional $H(t)$ satisfies the condition

$$\frac{dH}{dt} \leqslant 0 \tag{1.6}$$

This relationship is known as the *Boltzmann H-theorem*. The functional H with the opposite sign is identical to the entropy of the system, so that eq. (1.6) expresses the entropy increase law for the system under consideration.

Problem 1.3. Find the velocity distribution for the gas particles in thermodynamic equilibrium.

In this case $dH/dt = 0$, which is true if $\ln(ff_1/f'f_1') = 0$, that is, if $\ln f$ is an additive function of the integrals of motion. Using the laws of conservation of momentum and energy in collisions, we can write for the equilibrium distribution function

$$\ln f = C + C_1\mathbf{p} + C_2\varepsilon$$

whence

$$f = A\exp[-\alpha(\mathbf{v}-\mathbf{v}_0)^2]$$

where $\varepsilon = mv^2/2$ is the particle's energy, $\mathbf{p} = m\mathbf{v}$ is its momentum, and m is the particle's mass. The above equation contains three indefinite constants. The constant A can be found from the normalization condition $\int f(\mathbf{v})\,d\mathbf{v} = N$ for the distribution function, where N is the number density of the gas particles. Another constant is expressed through the velocity $\mathbf{v}_0$ of the directed motion of the gas particles. The third constant, α, can be found from the condition that in the reference system in which there is no directed motion of the particles the mean energy of the particles should be $(3/2)T$, where T is the gas temperature. We obtain

$$\frac{3}{2}T = \frac{\int \frac{m}{2}(\mathbf{v}-\mathbf{v}_0)^2\exp[-\alpha(\mathbf{v}-\mathbf{v}_0)^2]\,d\mathbf{v}}{\int \exp[-\alpha(\mathbf{v}-\mathbf{v}_0)]^2\,d\mathbf{v}}$$

$$= -\frac{m}{2}\frac{d}{d\alpha}\ln\int\exp[-\alpha(\mathbf{v}-\mathbf{v}_0)^2]\,d\mathbf{v} = -\frac{m}{2}\frac{d}{d\alpha}\ln\alpha^{-3/2} = \frac{3m}{4\alpha}$$

and, hence, $\alpha = m/2T$. Thus, the equilibrium velocity distribution function normalized for the gas density, which is known as the *Maxwellian distribution*, has the following form:

$$f(\mathbf{v}) = N\left(\frac{m}{2\pi T}\right)^{3/2}\exp\left[-\frac{m(\mathbf{v}-\mathbf{v}_0)^2}{2T}\right] \equiv N\varphi(v_x)\varphi(v_y)\varphi(v_z) \quad (1.7)$$

where

$$\varphi(v_x) = \left(\frac{m}{2\pi T}\right)^{1/2}\exp\left[-\frac{m(v_x-v_{0x})^2}{2T}\right], \quad \int_{-\infty}^{+\infty}\varphi(v_x)\,dv_x = 1 \quad (1.8)$$

The H-theorem yields the following important property of the distribution function $f(\mathbf{v}, t)$: in the absence of external fields the system tends to the equilibrium corresponding to the equilibrium distribution (1.7). Hence, the distribution function which is a solution of the Boltzmann equation (1.2) does not satisfy the principle of time reversibility. Indeed, let at $t = -T$ $(T \to \infty)$ the system be in a certain non-equilibrium state described by the distribution function $f_0(\mathbf{v})$. According to the H-theorem, at $t = T$ the system goes over to the equilibrium state described by the Maxwellian distribution (1.7). If, on the contrary, at $t = -T$ the system is in the equilibrium state, then at $t = T$, according to the H-theorem, the system remains in this state rather than pass over to the state described by the distribution function $f_0(\mathbf{v})$, as should be expected from reversibility of time (that is, the requirement that the system develops in the reverse direction when t is replaced by $-t$). Hence, the Boltzmann kinetic equation does not satisfy the principle of time reversibility, while the laws of mechanics from which this equation has been derived satisfy this principle.

Thus, we have found that the distribution function which is a solution of the Boltzmann equation cannot describe the system exactly. The distribution function is seen to contain a much smaller amount of information than is needed for the exact description of the system. Indeed, to describe the system exactly we have to specify the initial states for all the particles, that is, the number of parameters describing the system is proportional to the number of particles in it. But when we introduce a distribution function describing the probability that particles with given parameters are in a given space element, we thus perform averaging over the initial conditions. Therefore, the distribution function which is a solution of the Boltzmann equation (1.2) describes the most probable state of the system for given macroscopic parameters.

If the system consists of a large number of particles, the most probable state is practically the exact state, since in such a system fluctuations (deviations from the most probable state) are small. Moreover, it appears practically unfeasible to find the exact state of a system consisting of a large number of particles, since the number of parameters describing the system is very large. The number of parameters increases with the number of particles in the system. The distribution function satisfying eq. (1.2) has a smaller number of variables and this number does not depend on the number of particles of a given species in the system. Thus, description of a system with the Boltzmann equation (1.2) gives the most probable state of the system and is, therefore, accurate if the number of particles in the system is large. The advantage of this approach is that it yields the most important information on the development of a system by selecting a limited number of parameters describing it.

Problem 1.4. Find the number of particles in a given state for a system of a large number of particles in a thermodynamic equilibrium (the Boltzmann distribution).

Let a system consist of n particles in various states. The energy of a particle in the state k is ε_k. Then, if n_k is the number of particles in the state k and E is the total energy of the given closed system, we obtain the following relationships for the total number of particles and the total energy:

$$n = \sum_k n_k \quad \text{and} \quad E = \sum_k \varepsilon_k n_k \tag{1.9}$$

We have to find the distribution of particles over states. We shall assume that the number of particles in a given state is large ($n_k \gg 1$). Furthermore, we shall make use of the Boltzmann statistics in which the probability that a particle is in a given state is not affected by other particles present in this state. The probability of a given distribution of particles over the states $P(n_1, n_2, \ldots, n_k, \ldots)$ is proportional to the number of ways by which this distribution can be realized.

To find the number of such ways, first, from the total number of particles we shall select n_1 particles in state 1. This can be done by $C^n_{n_1}$ ways ($C^n_{n_1}$ is the number of combinations). Then we shall select n_2 particles in state 2 from the remaining $n - n_1$ particles, which can be done by $C^{n-n_1}_{n_2}$ ways. This procedure is continued until we find that the probability of a given distribution of particles over the states, which is proportional to the number of ways for realizing this distribution, is

$$P(n_1, n_2, \ldots, n_k, \ldots) = AC^n_{n_1}C^{n-n_1}_{n_2} \ldots C^n_{n_k} = A \sum_{i=1}^{k} \frac{n!}{\prod_i n_i!}$$

where A is the normalization constant. The probability $P(n_1, \ldots, n_k, \ldots)$ has a maximum at some values $n_k = \bar{n}_k$ of occupation numbers which are the most probable numbers of particles in given states. Let us use the condition for a maximum of the distribution function. If the numbers of particles in three states n_i, n_k and n_l are varied by amounts small in comparison with $\bar{n}_i$, $\bar{n}_k$ and $\bar{n}_l$ ($n_i = \bar{n}_i + \delta_i$, $n_k = \bar{n}_k - \delta_k$, $n_l = \bar{n}_l + \delta_l$), then in the first approximation in the power series in δ_i/n_i the probability P is not changed: $P(\ldots, \bar{n}_i, \bar{n}_k, \bar{n}_l, \ldots) = P(\ldots, \bar{n}_i + \delta_i, \bar{n}_k - \delta_k, \bar{n}_l + \delta_l, \ldots)$. Hence we obtain $[(n_i + \delta_i)! \approx n_i(n_i)^{\delta_i}]$

$$(\bar{n}_i)^{\delta_i}(\bar{n}_l)^{\delta_l} = (\bar{n}_k)^{\delta_k}$$

Furthermore, the conditions for balance of particles and balance of energy yield the following relationships:

$$\delta_k = \delta_i + \delta_l, \quad \varepsilon_k\delta_k = \varepsilon_i\delta_i + \varepsilon_l\delta_l$$

Using these relationships to eliminate δ_i from the above equation, we obtain

$$\left(\frac{\bar{n}_k}{\bar{n}_i}\right)^{\frac{1}{\varepsilon_k - \varepsilon_i}} = \left(\frac{\bar{n}_l}{\bar{n}_k}\right)^{\frac{1}{\varepsilon_l - \varepsilon_k}}$$

This equation shows that each of the relationships does not depend on the numbers of the states chosen, that is, it is constant for a given total number of particles and the energy of the system. To meet this condition, the dependence of $\bar{n}_k$ on the energy of this state must be expressed as

$$\bar{n}_k = B \exp(-\varepsilon_k/T) \tag{1.10}$$

where the normalization constant B and the temperature of the system T are found from conditions (1.9). Equation (1.10) is the equilibrium distribution of particles over states and is known as the *Boltzmann distribution*. A special case of this distribution is the Maxwellian distribution found in Problem 1.3. The Boltzmann distribution yields the following relationship between the densities of atoms N_i and N_k in given states in thermodynamic equilibrium:

$$\frac{N_i}{N_k} = \frac{g_i}{g_k} \exp\left(\frac{\varepsilon_k - \varepsilon_i}{T}\right) \tag{1.11}$$

(here g_i and g_k are the statistical weights of these states).

Problem 1.5. Derive the distribution function for the particles with half-integral spin if not more than one particle can be in each state (the *Fermi-Dirac distribution*).

If the probability that a particle is in a given state is not small, the Boltzmann distribution (1.10) is not applicable. The distribution of identical particles over states depends then on the statistics of these particles. We shall find such distribution if not more than one particle can be in each state.

Let the particles in the g_k states have the energy ε_k and $p(n_k)$ be the number of ways to place n_k particles in the states with the energy ε_k. The probability of a given distribution of particles over states is

$$P(n_1, n_2, \ldots, n_k, \ldots) = C \prod_m p(n_k)$$

where the normalization constant C does not depend on the form of distribution.

As follows from the solution of Problem 1.4, the function P is at a maximum for the most probable values of occupation numbers $n_k = \bar{n}_k$. In this case $\ln P$ also is at a maximum, and we obtain

$$\sum_k \frac{p'(\bar{n}_k)}{p(\bar{n}_k)} \delta n_k = 0$$

where $\delta n_k = n_k - \bar{n}_k$ is the deviation of the occupation number from the mean value ($|\delta n_k| \ll \bar{n}_k$). Furthermore, eq. (1.9) yields

$$\sum \delta n_k = 0, \quad \sum_k \varepsilon_k \, \delta n_k = 0$$

Multiply the second of the above equations by $-\mu/T$ and the third by $1/T$ (where μ and T are constants) and add up the three equations. The result is

$$\sum_k \delta n_k \left[\frac{p'(\bar{n}_k)}{p(\bar{n}_k)} - \frac{\mu}{T} + \frac{\varepsilon_k}{T} \right] = 0$$

Since δn_k can have arbitrary values, the above equation is satisfied if the term in the square brackets is zero. Hence we obtain the relationship for the most probable occupation numbers:

$$\frac{p'(\bar{n}_k)}{p(\bar{n}_k)} - \frac{\mu}{T} + \frac{\varepsilon_k}{T} = 0$$

In particular, we find that for the Fermi-Dirac distribution, when n_k particles are in g_k-states with the energy ε_k and not more than one particle is in each state ($n_k \leqslant g_k$), $p(n_k)$ is equal to the number of combinations of n_k things g_k at a time. Hence, we obtain

$$\ln p(n_k) = \ln (g_k!) - [\ln (n_k!) - \ln (g_k - n_k)!]$$

and, since $\ln (n!) \approx \int\limits_0^n \ln n \, dn$ for $n \gg 1$, we have

$$\frac{d \ln p(n_k)}{dn_k} = \ln \frac{n_k}{g_k - n_k}$$

and

$$\bar{n}_k = \frac{g_k}{\exp \left[\dfrac{\varepsilon_k - \mu}{T} \right] + 1}$$

On the average, $\left[\exp \left(\dfrac{\varepsilon_k - \mu}{T} \right) + 1 \right]^{-1}$ particles are in one state with the energy ε_k. The constants T (the temperature of the particles) and μ (the chemical potential of the distribution) can be found from the normalization conditions (1.9):

$$\sum_k g_k \left[\exp \left(\frac{\varepsilon_k - \mu}{T} \right) + 1 \right]^{-1} = n$$
$$\sum_k \varepsilon_k g_k \left[\exp \left(\frac{\varepsilon_k - \mu}{T} \right) + 1 \right]^{-1} = E \tag{1.12}$$

For example, let us calculate the chemical potential of the free Fermi particles with the mass m, whose number density is N. The energy of the particles in the range from ε to $\varepsilon + d\varepsilon$ (here $\varepsilon = p^2/2m$) corresponds to

$\int \frac{d\mathbf{p}\, d\mathbf{r}}{(2\pi\hbar)^3} = V\frac{m^{3/2}\varepsilon^{1/2}\, d\varepsilon}{\sqrt{2}\pi^2\hbar^3}$ states, where V is the volume of the system under consideration. When the temperature tends to zero, the Fermi particles occupy the lowest states. Let us define the Fermi energy ε_F in the following way: at zero temperature all the states with the energy $\varepsilon \leqslant \varepsilon_F$ are occupied and all the states with higher energies are empty. Then the Fermi energy can be found from the relationship

$$V\frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int\limits_0^{\varepsilon_F} \varepsilon^{1/2}\, d\varepsilon = NV$$

which yields

$$\varepsilon_F = \frac{\pi\hbar^2}{m}\left(\frac{9\pi N^2}{2}\right)^{1/3}.$$

Let us analyze now the Fermi-Dirac distribution. It can be seen that at the zero temperature $\exp[(\varepsilon_k - \mu)/T]$ is zero for $\varepsilon_k < \mu$ and infinity for $\varepsilon_k > \mu$. Thus, the chemical potential at the zero temperature is identical to the Fermi energy defined above.

Problem 1.6. Derive the Boltzmann distribution and the Maxwellian distribution from the Fermi-Dirac distribution.

The Boltzmann distribution is the extreme case of the Fermi-Dirac distribution (for the particles with half-integral spin) when the probability of finding the particle in a certain state is small. Let us consider a gas of free Fermi particles with mass m and number density N at temperature T. Assume that the system occupies volume V. The number of states in the elementary phase space volume is $d\mathbf{p}\, d\mathbf{r}/(2\pi\hbar)^3$, where $d\mathbf{r}$ and $d\mathbf{p}$ are the elementary vectors in the coordinate space and the momentum space, respectively, and the particle's energy is $\varepsilon = p^2/2m$. Hence, the number of states corresponding to the energy range from ε to $\varepsilon + d\varepsilon$ is

$$\int \frac{d\mathbf{p}\, d\mathbf{r}}{(2\pi\hbar)^3} = V\frac{m^{3/2}\varepsilon^{1/2}\, d\varepsilon}{\sqrt{2}\pi^2\hbar^3}$$

Since the total number of particles in the volume V is NV and the characteristic energy is of the order of T, the following condition must be satisfied for the Boltzmann distribution to be valid (the probability of finding a particle in a given state to be low):

$$\frac{1}{N}\left(\frac{mT}{\hbar^2}\right)^{3/2} \gg 1$$

In this case the normalization condition for the Fermi-Dirac distribution is

$$\sum_k \frac{g_k}{\left[\exp\left(\frac{\varepsilon_k-\mu}{T}\right)+1\right]} \equiv \int \frac{d\mathbf{p}\,d\mathbf{r}}{(2\pi\hbar)^3\left[\exp\left(\frac{\varepsilon-\mu}{T}\right)+1\right]} = NV$$

Using the expression for the density of states and a new variable $x = \varepsilon/T$, we can rewrite the above equation as

$$\int_0^\infty \frac{x^{1/2}\,dx}{\left[\exp\left(x-\frac{\mu}{T}\right)+1\right]} = \sqrt{2}\pi^2 N\left(\frac{\hbar^2}{mT}\right)^{3/2}$$

In our approximation the right-hand side of this equation is much smaller than unity. Since the main contribution to the integral is made by the values of $x \sim 1$, the above condition implies $\exp(-\mu/T) \gg 1$. Hence we obtain

$$\exp(\mu/T) = \left(\frac{2\pi\hbar^2}{mT}\right)^{3/2} N \ll 1$$

which justifies the above assumption.

Using the above relationship for μ, we find that the probability of finding a particle in the state with the energy ε_k is

$$W_k = \left[\exp\left(\frac{\varepsilon_k-\mu}{T}\right)+1\right]^{-1} \approx \left(\frac{2\pi\hbar^2}{mT}\right)^{3/2} N\exp\left(-\frac{\varepsilon_k}{T}\right)$$

Hence, the number of particles in the volume V with the energies in the range from ε to $\varepsilon + d\varepsilon$ is

$$\bar{n}_k = \frac{g_k}{\exp\left(\frac{\varepsilon_k-\mu}{T}\right)+1} = \int \frac{d\mathbf{p}\,d\mathbf{r}}{(2\pi\hbar)^3}\exp\left(-\frac{\varepsilon}{T}+\frac{\mu}{T}\right)$$

$$= \frac{NV\,dp_x\,dp_y\,dp_z\exp\left(-\frac{\varepsilon}{T}\right)}{(2\pi mT)^{3/2}}$$

This equation is the sought Maxwellian distribution.

Problem 1.7. Derive the equation of state for an ideal gas.

The equation of state for a gas is a relationship between its parameters (temperature and pressure). Let us find the pressure in a gas, that is, the force acting on unit surface area per unit time. Let us find the forces acting on an imaginary elementary surface area in the gas from both sides.

Reflection of atoms from this surface is elastic, so that the tangential component of their velocity is conserved and the normal component is reversed. Let the x-axis be normal to the surface. Then the momentum of an atom reflected from the surface is changed by $2mv_x$. The intensity of the flux of atoms incident on the imaginary surface from the left is Nv_x. Then the pressure which is equal to the variation of the momentum of the imaginary surface per unit time per unit area due to atoms incident from the left is

$$P = \int_{v_x > 0} 2mv_x \cdot Nv_x f(\mathbf{v})\, d\mathbf{v}$$

Here the velocity distribution function $f(\mathbf{v})$ is normalized to unity. The gas pressure due to atoms incident from the right is

$$P = \int_{v_x < 0} 2mv_x \cdot Nv_x f(\mathbf{v})\, d\mathbf{v}$$

Since $f(\mathbf{v})$ is even with respect to v_x, the two definitions of pressure are identical *.

Hence we obtain

$$P = \int_{v_x > 0} 2mv_x^2 Nf(\mathbf{v})\, d\mathbf{v} = N\int mv_x^2 f(\mathbf{v})\, d\mathbf{v} = N\langle mv_x^2\rangle$$

where the angle brackets mean averaging over the distribution function. If the distribution function is isotropic, as is the case at thermodynamic equilibrium in the coordinate system under consideration, we have

$$\langle v_x^2\rangle = \langle v_y^2\rangle = \langle v_z^2\rangle = \frac{1}{3}\langle v^2\rangle$$

Since the gas temperature is defined as $T = 1/3\langle mv^2\rangle$, we obtain the equation of state

$$P = NT$$

Problem 1.8. Derive from the Boltzmann kinetic equation the principal macroscopic transport equations.

Multiply the Boltzmann equation (1.5) written in the form

$$\frac{\partial f_1}{\partial t} + \mathbf{v}_1 \nabla f_1 + \frac{\mathbf{F}}{m_1}\frac{\partial f_1}{\partial \mathbf{v}_1} = I_c(f_1)$$

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* If the average velocity of atoms differs from zero, the pressure should be determined in the coordinate system in which the gas as a whole is at rest. The gas temperature is determined in the same coordinate system.

by an arbitrary function Ψ_1 of the velocity of a given species and integrate the result over velocities. We obtain

$$\int \Psi_1 \frac{\partial f_1}{\partial t}\, d\mathbf{v}_1 = \frac{\partial}{\partial t}\int \Psi_1 f_1\, d\mathbf{v}_1 - \int f_1 \frac{\partial \Psi_1}{\partial t}\, d\mathbf{v}_1 = \frac{\partial}{\partial t}(N_1\langle\Psi_1\rangle) - N_1\left\langle \frac{\partial\Psi_1}{\partial t}\right\rangle$$

The angle brackets denote averaging over velocities and N_1 is the number density of the particles of a given species. Further we obtain

$$\int \Psi_1 v_{1x}\frac{\partial f_1}{\partial x}\, d\mathbf{v}_1 = \frac{\partial}{\partial x}\int \Psi_1 \mathbf{v}_{1x} f_1\, d\mathbf{v}_1 - \int f_1 v_{1x}\frac{\partial \Psi_1}{\partial x}\, d\mathbf{v}_1$$

$$= \frac{\partial}{\partial x}(N_1\langle\Psi_1 v_{1x}\rangle) - N_1\left\langle v_{1x}\frac{\partial\Psi_1}{\partial x}\right\rangle$$

$$\int \Psi_1\frac{\partial f_1}{\partial v_{1x}}\, d\mathbf{v}_1 = \int \frac{\partial \Psi_1 f_1}{\partial v_{1x}}\, d\mathbf{v}_1 - \int f_1 \frac{\partial\Psi_1}{\partial v_{1x}}\, d\mathbf{v}_1 = -N_1\left\langle\frac{\partial \Psi_1}{\partial v_{1x}}\right\rangle$$

The result is

$$\frac{\partial(N_1\langle\Psi_1\rangle)}{\partial t} + \operatorname{div}(N_1\langle\Psi_1\mathbf{v}_1\rangle)$$

$$- N_1\left[\frac{\partial\langle\Psi_1\rangle}{\partial t} + \langle\mathbf{v}_1 \operatorname{grad}\Psi_1\rangle + \frac{\mathbf{F}_1}{m_1}\left\langle\frac{\partial\Psi_1}{\partial\mathbf{v}_1}\right\rangle\right] = \langle I_c\Psi_1\rangle \qquad (1.13)$$

Let us average this equation over the internal degrees of freedom of a given species. This does not change the form of the equation if the angle brackets are assumed to denote averaging over velocities and internal degrees of freedom. Equation (1.13) is known as the *Enskog general equation.*

Let us prove an important theorem which can be used for significant simplification of the right-hand side of eq. (1.13). Let Ψ be an integral of motion, so that we have for two colliding particles of the ith species

$$\Psi_i(\mathbf{v}_1) + \Psi_i(\mathbf{v}_2) = \Psi_i(\mathbf{v}_1') + \Psi_i(\mathbf{v}_2') \qquad (1.14)$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocities of particles before the collision and $\mathbf{v}_1'$ and $\mathbf{v}_2'$ are the velocities after the collision. Then the following equation can be written for the collision integral for the particles of the same species:

$$\langle I_c\Psi_i\rangle = 0 \qquad (1.15)$$

Indeed, the definition of the collision integral (1.4) yields

$$\langle I_c\Psi_1\rangle = \int \Psi(\mathbf{v}_1)(f_1'f_2' - f_1f_2)W(\mathbf{v}_1,\mathbf{v}_2 \to \mathbf{v}_1',\mathbf{v}_2')\, d\mathbf{v}_1\, d\mathbf{v}_2\, d\mathbf{v}_1'\, d\mathbf{v}_2'$$

Making the replacement $\mathbf{v}_1 \to \mathbf{v}_2$, $\mathbf{v}_2 \to \mathbf{v}_1$ in the integrand, we obtain

$$\langle I_c \Psi_1 \rangle = \frac{1}{2} \int [\Psi(\mathbf{v}_1) + \Psi(\mathbf{v}_2)](f_1' f_2' - f_1 f_2) W \, d\mathbf{v}_1 \, d\mathbf{v}_2 \, d\mathbf{v}_1' \, d\mathbf{v}_2'$$

Further, since $W = W(\mathbf{v}_1, \mathbf{v}_2 \to \mathbf{v}_1', \mathbf{v}_2') = W(\mathbf{v}_1', \mathbf{v}_2' \to \mathbf{v}_1, \mathbf{v}_2)$, we can make the replacement $\mathbf{v}_{1,2} \to \mathbf{v}_{1,2}'$, $\mathbf{v}_{1,2}' \to \mathbf{v}_{1,2}$ in the integrand and obtain

$$\langle I_c \Psi_1 \rangle = \frac{1}{4} \int [\Psi(\mathbf{v}_1) + \Psi(\mathbf{v}_2) - \Psi(\mathbf{v}_1') - \Psi(\mathbf{v}_2')] \times (f_1' f_2' - f_1 f_2) W \, d\mathbf{v}_1 \, d\mathbf{v}_2 \, d\mathbf{v}_1' \, d\mathbf{v}_2'$$

This equation and eq. (1.14) yield eq. (1.15).

Let us derive, using eq. (1.13), the principal transport equation for the gas consisting of one species. Assume $\Psi_1 = 1$ in eq. (1.13). This yields the continuity equation

$$\frac{\partial N}{\partial t} + \operatorname{div} (N\mathbf{v}_0) = 0 \tag{1.16}$$

where $\mathbf{v}_0$ is the mean velocity of atoms at a given moment in a given point in space.

Let us set $\Psi_1 = m\mathbf{v}$ in eq. (1.13) and make use of the pressure tensor

$$P_{jk} = Nm \langle (v_j - v_{0j})(v_k - v_{0k}) \rangle \tag{1.17}$$

where the subscripts j, k denote the respective components of the velocity vector. If the subscripts are identical, P_{jj} is the gas pressure in a given direction, as follows from the results of Problem 1.7. Using the pressure tensor (1.17), we can find the jth component of the momentum transport equation:

$$\frac{\partial}{\partial t} (mNv_{0j}) + \frac{\partial P_{jk}}{\partial x_k} + \frac{\partial}{\partial x_k} (mNv_{0j}v_{0k}) - NF_j = 0$$

Summation over the subscripts repeated twice is implied in this equation. Using the continuity equation $\frac{\partial N}{\partial t} + \frac{\partial}{\partial x_k} (Nv_{0k}) = 0$, we derive the motion equation for the gas (the *Euler equation*)

$$\frac{\partial v_{0j}}{\partial t} + v_{0k} \frac{\partial v_{0j}}{\partial x_k} + \frac{1}{mN} \frac{\partial P_{jk}}{\partial x_k} - \frac{F_j}{m} = 0 \tag{1.18}$$

To derive the energy transport equation we shall first define the heat flux intensity

$$q_j = \frac{1}{2} mN \langle (\mathbf{v} - \mathbf{v}_0)^2 (v_j - v_{0j}) \rangle \tag{1.19}$$

Further, setting $\Psi = mv^2/2$ in the transport equation (1.13), we obtain

$$\frac{\partial}{\partial t}\left(N\frac{mv_0^2}{2}\right)+\frac{\partial}{\partial t}\left(N\left\langle\frac{m(\mathbf{v}-\mathbf{v}_0)^2}{2}\right\rangle\right)+\frac{\partial}{\partial x_k}\left(N\frac{mv_0^2}{2}v_{0k}\right)$$
$$+\frac{\partial}{\partial x_k}v_{0j}P_{jk}+\frac{\partial}{\partial x_k}\left[Nv_{0k}\left\langle\frac{m(\mathbf{v}-\mathbf{v}_0)^2}{2}\right\rangle\right]+\frac{\partial q_k}{\partial x_k}-NF_kv_{0k}=0$$

This equation can be transformed into

$$\frac{mv_0^2}{2}\left\{\frac{\partial N}{\partial t}+\frac{\partial Nv_{0k}}{\partial x_k}\right\}+Nmv_{0j}\left\{\frac{\partial v_{0j}}{\partial t}+v_{0k}\frac{\partial v_{0j}}{\partial x_k}+\frac{1}{Nm}\frac{\partial P_{jk}}{\partial x_k}-\frac{F_j}{m}\right\}$$
$$+\frac{\partial}{\partial t}\left[N\left\langle\frac{m(\mathbf{v}-\mathbf{v}_0)^2}{2}\right\rangle\right]+\frac{\partial}{\partial x_k}\left[Nv_{0k}\left\langle\frac{m(\mathbf{v}-\mathbf{v}_0)^2}{2}\right\rangle\right]+P_{jk}\frac{\partial v_{0j}}{\partial x_k}+\frac{\partial q_k}{\partial x_k}=0$$

According to the equations of transport of mass (1.16) and momentum (1.14), the terms in braces are zero. Using the gas temperature $T=\left\langle\frac{m(\mathbf{v}-\mathbf{v}_0)^2}{3}\right\rangle$, we obtain the following equation for energy transport (the *heat conduction equation*):

$$\frac{3}{2}N\left(\frac{\partial T}{\partial t}+\mathbf{v}_0\nabla T\right)=-P_{jk}\frac{\partial v_{0j}}{\partial x_k}-\frac{\partial q_k}{\partial x_k} \tag{1.20}$$

Problem 1.9. Derive the Euler equation for inviscid gas.

In an ideal inviscid gas no correlation exists between motions in various directions. Therefore, the pressure tensor (1.17) can be written as

$$P_{jk}=\delta_{jk}Nm\langle(v_k-v_{0k})^2\rangle$$

where N is the number density of gas particles, m is their mass, and $\mathbf{v}_0$ is the mean velocity of the gas. Note that in the reference frame in which the gas is at rest the distribution function for particles is spherically symmetric. This means that pressure in all directions is the same. Using the definition of pressure

$$p=Nm\langle(v_x-v_{0x})^2\rangle=Nm\langle(v_y-v_{0y})^2\rangle=Nm\langle(v_z-v_{0z})^2\rangle$$

we find the pressure tensor

$$P_{jk}=p\delta_{jk} \tag{1.21}$$

Substituting eq. (1.21) into the Euler equation (1.18), we can rewrite it as

$$\frac{\partial\mathbf{v}_0}{\partial t}+(\mathbf{v}_0\nabla)\mathbf{v}_0+\frac{1}{Nm}\nabla p-\frac{\mathbf{F}}{m}=0$$

Problem 1.10. Derive the equations of transport of momentum and energy for ions travelling in a gas in a constant electric field.

The Boltzmann equation (1.5) for the ion distribution function, taking into account elastic scattering of ions by atoms, is

$$\frac{eE}{M}\frac{\partial f}{\partial v_x} = N\int [f(\mathbf{v}')\varphi(v_1') - f(\mathbf{v})\varphi(v_1)]\,|\mathbf{v}-\mathbf{v}_1|\,d\sigma\,d\mathbf{v}_1$$

where E is the electric field strength, M is the ion mass, $\mathbf{v}$ and $\mathbf{v}_1$ are the velocities of the ion and the atom before the collision, $\mathbf{v}'$ and $\mathbf{v}_1'$ are their velocities after the collision, $f(\mathbf{v})$ is the ion distribution function, $\varphi(v)$ is the Maxwellian atom distribution function, and N is the number density of atoms. The distribution functions $f(\mathbf{v})$ and $\varphi(v)$ are normalized to unity.

Multiply the Boltzmann equation by Mv_x and integrate it over the ion velocities, obtaining

$$-eE = \int M(v_x' - v_x)f(\mathbf{v})\varphi(v_1)N\,|\mathbf{v}-\mathbf{v}_1|\,d\sigma\,d\mathbf{v}\,d\mathbf{v}_1$$

The left-hand side of this equation is the momentum given by the electric field to the ion per unit time, and the right-hand side is the momentum transferred to the gas atoms from the ion per unit time. Let us simplify this equation. We can write

$$\mathbf{v} = \mathbf{v}_c + \frac{m}{m+M}\mathbf{g}, \quad \mathbf{v}' = \mathbf{v}_c + \frac{m}{m+M}\mathbf{g}', \quad \mathbf{v}-\mathbf{v}' = \frac{m}{m+M}(\mathbf{g}-\mathbf{g}')$$

where m is the atom's mass, $\mathbf{v}_c$ is the velocity of the centre of inertia of the colliding atom and ion, and $\mathbf{g}$ and $\mathbf{g}'$ are their relative velocities before and after the collision (for elastic collision $g = g'$).

Denote the polar angle of scattering as θ and the azimuthal angle as Φ. Then $\mathbf{g}' = \mathbf{g}\cos\theta + \mathbf{k}g\sin\theta$, where $\mathbf{k}$ is the unit vector perpendicular to $\mathbf{g}$. Since $\int_0^{2\pi} \mathbf{k}\,d\Phi = 0$, then

$$\frac{1}{2\pi}\int_0^{2\pi}(v_x - v_x')\,d\Phi = \frac{m}{m+M}g_x(1-\cos\theta)$$

and

$$eE = \mu N\int f(\mathbf{v})\varphi(v_1)gg_x\sigma^*(g)\,d\mathbf{v}\,d\mathbf{v}_1 \tag{1.22}$$

where $\mu = mM/(m+M)$ is the reduced mass of the ion and atom, and $\sigma^*(g) = \int(1-\cos\theta)\,d\sigma$ is the transport cross section for scattering of the ion on the atom.

To derive the equation for energy transfer multiply the kinetic equation by $Mv^2/2$ and integrate the result over the ion velocities. We obtain

$$eEw = \int f(\mathbf{v})\varphi(v_1)\frac{M(v^2 - v'^2)}{2} Ng\, d\sigma\, d\mathbf{v}_1\, d\mathbf{v}$$

where w is the drift ion velocity. Since $\mathbf{v} = \mathbf{v}_c + \frac{m}{m+M}\mathbf{g}$ and $\mathbf{v}' = \mathbf{v}_c + \frac{m}{m+M}\mathbf{g}'$, we have $v^2 - v'^2 = \frac{2m}{m+M}\mathbf{v}_c(\mathbf{g} - \mathbf{g}')$. Since

$$\frac{1}{2\pi}\int_0^{2\pi}(\mathbf{g} - \mathbf{g}')\, d\Phi = \mathbf{g}(1 - \cos\theta)$$

we obtain the equation

$$eEw = \int f(\mathbf{v})\varphi(v_1)\mathbf{v}_c\mathbf{g}Ng\sigma^*\, d\mathbf{v}\, d\mathbf{v}_1 \tag{1.23}$$

where $\mathbf{v}_c = (M\mathbf{v} + m\mathbf{v}_1)/(m + M)$ and $\mathbf{g} = \mathbf{v} - \mathbf{v}_1$.

Equation (1.22) describes the momentum transfer and eq. (1.23) describes the energy transfer between the ion, atoms and the electric field. The left-hand side of eq. (1.22) is the momentum transferred from the field to the ion per unit time, and the right-hand side is the momentum transferred from the ion to the gas atoms per unit time. The left-hand side of eq. (1.23) is the energy transferred from the field to the ion per unit time, and the right-hand side is the energy transferred from the ion to the gas atoms per unit time.

Problem 1.11. Find the mean velocity of ions travelling in a gas in a constant electric field. The density of ions is much lower than the density of the gas particles and its variation along the mean free path of the ion is small; the frequency of collisions between the ions and the gas particles is independent of their relative velocity.

The kinetic equation for the ion velocity distribution function f ignoring the inelastic collisions is

$$\frac{\partial f}{\partial t} + \mathbf{v}\nabla f + \frac{e\mathbf{E}}{M}\frac{\partial f}{\partial \mathbf{v}} = I_c(f) \tag{1.24}$$

where $\mathbf{v}$ is the ion's velocity, M is its mass, $\mathbf{E}$ is the electric field strength, and I_c is the collision integral for collisions between the ions and the gas particles. Since the characteristic length for variation of the ion number density is large in comparison with the mean free path, the distribution function is proportional to the ion number density at a given point in space. Assuming that the ion number density N_i determines the coordinate dependence of the distribution function, we can transform the kinetic equation into

$$(v_x - \langle v_x \rangle) \frac{\partial \ln N_i}{\partial x} f + \frac{e\mathbf{E}}{M} \frac{\partial f}{\partial \mathbf{v}} = I_c(f) \tag{1.25}$$

where x is the direction along which the ion number density varies and $\langle v_x \rangle$ is the mean ion's velocity along this direction.

Now multiply the equation by Mv_x and integrate the result over the ion velocities. This has been done in Problem 1.10 for all the equation terms except the first one. We obtain

$$(\langle Mv_x^2 \rangle - M\langle v_x \rangle^2) \frac{\partial \ln N_i}{\partial x} - e\mathbf{E}\mathbf{n} = -\mu N \langle g g_x \sigma^*(g) \rangle$$

where $\mathbf{n}$ is the unit vector along the x-axis, μ is the reduced mass of the ion and the gas particle, g is the relative velocity of collision, and $\sigma^*(g)$ is the diffusion cross section for scattering of the ion on the gas particle.

In this problem the frequency of collisions of the ion with the gas particles $\nu = Ng\sigma^*(g)$ is assumed to be independent of the relative collision velocity. Hence we obtain the following ~~relat~~ionship, taking into account that the mean velocity of the gas particles is zero:

$$\mu\nu\langle v_x \rangle = -(\langle Mv_x^2 \rangle - M\langle v_x \rangle^2) \frac{\partial \ln N_i}{\partial x} + eE_x$$

In particular, if the electric field strength and the gas density gradient are parallel, we have

$$\langle v_x \rangle = -\frac{M(\langle v_x^2 \rangle - \langle v_x \rangle^2)}{\mu\nu} \frac{\partial \ln N_i}{\partial x} + \frac{eE}{\mu\nu} \tag{1.26a}$$

When these vectors are perpendicular, we have

$$\langle v_x \rangle = -\frac{M\langle v_x^2 \rangle}{\mu\nu} \frac{\partial \ln N_i}{\partial x} \tag{1.26b}$$

§ 1.2. TRANSPORT PHENOMENA IN GASES

Let the macroscopic gas parameters (gas temperature, velocity of directed motion, density of a given species) vary spatially. After a sufficiently long time these parameters will be levelled off spatially owing to migration of the gas particles. The equilibrium is established by energy transfer from the regions with higher temperature to the regions with lower temperature, by transfer of momentum if there are gradients of velocity of the directed gas motion, and by transfer of a given species if its density varies in the gas.

The transport phenomena can be meaningfully analyzed if the macroscopic gas parameters vary noticeably over distances L which are considerably larger than the mean free path λ of the gas particles:

$$\lambda/L \ll 1 \tag{1.27}$$

When the gradients of the number density of a given species, the temperature and the translational velocity of particles are small, the resulting fluxes of particles, energy and momentum are proportional to these gradients. The coefficients of proportionality between the fluxes and the gradients are known as the *kinetic transport coefficients.* For instance, the *diffusion coefficient* $\mathscr{D}$ is defined as the proportionality factor between the particle flux $\mathbf{j}$ and the gradient of concentration c of a given species:

$$\mathbf{j} = -\mathscr{D} N \nabla c$$

Here N is the total number density of particles. If the concentration of a given species is low ($c \ll 1$), that is, if this species is an impurity in the gas, the flux of the particles of this species can be written as

$$\mathbf{j} = -\mathscr{D} \nabla N_1$$

where N_1 is the number density of the given species.

The *heat conductivity* $\varkappa$ is defined as the proportionality factor between the heat flux $\mathbf{q}$ and the temperature gradient:

$$\mathbf{q} = -\varkappa \nabla T$$

The *viscosity coefficient* η is the proportionality factor between the friction force acting on a moving gas and the gradient of the translational velocity of the gas. If the gas velocity $\mathbf{v}_0$ is parallel to the x-axis and varies in the z direction (see Fig. 1.2), the friction force parallel to the x-axis tends to equalize the velocities of the adjacent gas layers. This force is proportional to $\partial v_{0x}/\partial z$ and acts on an element xy perpendicular to the velocity gradient. If we define the force acting on a unit area of the layer as F, we can write

$$F = -\eta \frac{\partial v_{0x}}{\partial z}$$

(see also Problem 1.20).

Problem 1.12. Derive the kinetic equation describing transport phenomena in a monoatomic gas, assuming only elastic collisions of the gas particles.

The equilibrium in a gas is established in time during which the gas particles travel a distance of the order of L. According to condition (1.27), this time is much greater than the time of collision (the collision interval) between gas particles (the characteristic time between two subsequent collisions of the test particle with gas particles). Since the local equilibrium at a given point is established during the time of the order of the collision interval, the local equilibrium can be assumed to exist at each point during the transport processes, so that the velocity distribution function at a given point is

$$f^{(0)} = N(\mathbf{r})\left[\frac{m}{2\pi T(\mathbf{r})}\right]^{3/2}\exp\left[-\frac{m[(\mathbf{v}-\mathbf{v}_0(\mathbf{r})]^2}{2T(\mathbf{r})}\right] \tag{1.28}$$

Here $N(\mathbf{r})$ and $T(\mathbf{r})$ are the density and the temperature of the particles at a given point and $\mathbf{v}_0(\mathbf{r})$ is the velocity of the directed motion of the gas.

Let us expand the velocity distribution function in powers of the small parameter (1.27):

$$f = f^{(0)} + f^{(1)} + \ldots$$

Let the density of the gas particles, the gas temperature and the velocity of the directed gas motion be determined by the zero-order distribution function (1.28). Then we have for $i \neq 0$

$$\int f^{(i)}\,d\mathbf{v} = 0, \tag{1.29a}$$

$$\int f^{(i)}\mathbf{v}\,d\mathbf{v} = 0, \tag{1.29b}$$

$$\int f^{(i)}(\mathbf{v}-\mathbf{v}_0)^2\,d\mathbf{v} = 0 \tag{1.29c}$$

Let us express the correction term for the distribution function as $f^{(1)} = f^{(0)}\theta$, where θ describes the deviation of the distribution function from the Maxwell distribution. Substitute this relationship into the kinetic equation. This approximation describes the case when the equilibrium between the particles in a given spatial region is established in a much shorter time than the equilibrium between the particles in this region and the environment. Therefore, in the zero-order approximation we can ignore nonuniformities and write the kinetic equation in the following form:

$$I_c(f^{(0)}) = 0$$

Hence we obtain $f_1^{(0)}f_2^{(0)} = f_1^{(0)\prime}f_2^{(0)\prime}$ and eq. (1.28) for the velocity distribution function. Taking the next term in the expansion in powers of λ/L, we obtain the following kinetic equation:

$$\mathbf{v}_1\nabla f_1^{(0)} + \frac{\mathbf{F}}{m}\frac{\partial f_1^{(0)}}{\partial \mathbf{v}_1} = \int f_1^{(0)}f_2^{(0)}(\theta_1' + \theta_2' - \theta_1 - \theta_2)\,|\mathbf{v}_1 - \mathbf{v}_2|\,d\sigma\,d\mathbf{v}_2 \tag{1.30}$$

Problem 1.13. Calculate the heat conductivity of a monoatomic gas in the absence of external fields and steady-state gas flows.

Under the conditions of this problem the left-hand side of the kinetic equation contains only the term $\mathbf{v}\nabla f^{(0)}$ and, since only the temperature gradients differ from zero, the kinetic equation (1.30) has the following form:

$$f_1^{(0)}\left(\frac{mv_1^2}{2T}-\frac{5}{2}\right)\mathbf{v}_1\nabla \ln T = \int f_1^{(0)}f_2^{(0)}(\theta_1'+\theta_2'-\theta_1-\theta_2)\,|\mathbf{v}_1-\mathbf{v}_2|\,d\sigma\,d\mathbf{v}_2 \quad (1.31)$$

Here we assume that the gas pressure $p = NT$ is constant throughout the gas volume. The solution of the linear nonhomogeneous integral equation (1.31) with the conditions (1.29 a, b and c) can be written as $\theta = -\mathbf{A}\nabla \ln T$. For small gradients the vector $\mathbf{A}$ is independent of the gradients of temperature, density and velocity and can be constructed from the vectors which are observed in absence of gradients of these macroscopic parameters. The only such vector is the atom velocity vector $\mathbf{v}$, so that writing $\mathbf{A} = A(v)\mathbf{v}$, we obtain $\theta = -\mathbf{v}\nabla \ln TA(v)$. Let us write down eq. (1.31) in terms of dimensionless variables using the reduced velocity $\mathbf{u} = \sqrt{m/2T}\,\mathbf{v}$, where m is the particle's mass. We obtain

$$f_1^{(0)}\left(u_1^2-\frac{5}{2}\right)\mathbf{u}_1\left(\frac{2T}{m}\right)^{1/2} = \int f_1^{(0)}f_2^{(0)}[A_1'\mathbf{u}_1'+A_2'\mathbf{u}_2'-A_1\mathbf{u}_1-A_2\mathbf{u}_2]\,|\mathbf{u}_1-\mathbf{u}_2|\,d\sigma\,d\mathbf{u}_2 \quad (1.32)$$

where

$$f^{(1)} = -f^{(0)}\mathbf{v}\nabla \ln TA(v) \quad \text{and} \quad \int f^{(0)}\,d\mathbf{u} = N$$

It can be seen that to solve the problem we have to find the scalar quantity $A(v)$ which is a solution of the nonhomogeneous integral equation (1.32). This can be conveniently done with the approximate Chapman-Enskog method using expansion of the function $A(v)$ in Sonin polynomials:

$$A(u) = \sum_{m=0}^{\infty} a_m S_n^m(u^2)$$

The Sonin polynomials are

$$S_n^m(x) = \sum_{k=0}^{\infty}(-x)^k\frac{\Gamma(n+m+1)}{\Gamma(n+k+1)k!(m-k)!}, \quad S_n^0 = 1, \quad S_n^1 = n+1-x$$

and they differ from the generalized Laguerre polynomials only in a constant factor:

$$S_n^m(x) = \frac{(-1)^n}{\Gamma(m+n+1)} L_n^m(x)$$

The condition of orthogonality for the Sonin polynomials is

$$\int_0^\infty S_n^m(x) S_n^p(x) \exp(-x) x^n \, dx = \frac{\Gamma(n+m+1)}{m!} \delta_{mp}$$

where δ_{mp} is the Kronecker delta ($\delta_{mp} = 0$ for $m \neq p$; $\delta_{mp} = 1$ for $m = p$).

Expansion of the scalar quantities describing the transport phenomena in the Sonin polynomials is convenient since conditions (1.29) are automatically satisfied for each term of such expansion. For instance, for analyzing heat conduction it is convenient to take $n = 3/2$. In this case $a_0 = 0$ and condition (1.29b) is satisfied for all other terms of expansion of A in the Sonin polynomials:

$$A(u) = \sum_{m=1}^{\infty} a_m S_{3/2}^m(u^2)$$

The following notation will be used below:

$$\alpha_m = \int f^{(0)} \left(u^2 - \frac{5}{2} \right) S_{3/2}^m(u^2) u^2 \, d\mathbf{u}$$

$$= \frac{2N}{\sqrt{\pi}} \int_0^\infty \exp(-x) S_{3/2}^1(x) S_{3/2}^m(x) x^{3/2} \, dx = \frac{15}{4} N \delta_{m1}$$

$$\beta_{km} = \int f_1^{(0)} f_2^{(0)} [\mathbf{u}_1 S_{3/2}^k(u_1^2) + \mathbf{u}_2 S_{3/2}^k(u_2^2) - \mathbf{u}_1' S_{3/2}^k(u_1'^2)$$

$$- \mathbf{u}_2' S_{3/2}^k(u_2'^2)] \mathbf{u}_1 S_{3/2}^m(u_1^2) \, |\mathbf{u}_1 - \mathbf{u}_2| \, d\sigma \, d\mathbf{u}_1 \, d\mathbf{u}_2 \sqrt{\frac{2T}{m}},$$

Multiplying eq. (1.32) by $\mathbf{u}_1 S_{3/2}^m(u_1^2)$ and integrating the result over $d\mathbf{u}_1$, we obtain the system of equations for the coefficients a_k:

$$\sum_{k=1}^{\infty} a_k \beta_{km} = \frac{15}{4} N \delta_{m1}, \qquad m = 1, 2, \ldots$$

Solving this infinite system of equations, we can find the coefficients a_k.

Let us express the heat conductivity of the gas in terms of a_k. The intensity of the heat flux (energy flux density) is

$$\mathbf{q} = \int \frac{mv^2}{2}\mathbf{v} f\, d\mathbf{v} = \int \frac{mv^2}{2}\mathbf{v}\theta f^{(0)}\, d\mathbf{v} = -\frac{2T^2}{m}\int u^2\mathbf{u}(\mathbf{u}\nabla \ln T)Af^{(0)}\, d\mathbf{u}$$

$$= -\frac{2T^2}{3m}\int u^4 \nabla \ln T \sum_{m=1}^{\infty} a_m S^m_{3/2}(u^2) f^{(0)}\, d\mathbf{u} = -\varkappa \nabla T \qquad (1.33)$$

where the heat conductivity is

$$\varkappa = \frac{4TN}{3\sqrt{\pi m}}\int_0^{\infty} \exp(-z) z^5\, dz \sum_{m=1}^{\infty} a_m S^m_{3/2}(z) = \frac{5}{2} a_1 \frac{T}{m} N$$

From the system of equations (1.33) we obtain

$$a_1 = \lim_{n\to\infty} \frac{\frac{15}{4} N \begin{vmatrix} \beta_{22} & \beta_{23} \cdots \beta_{2n} \\ \beta_{32} & \beta_{33} \cdots \beta_{3n} \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot \\ \beta_{n2} & \beta_{n3} \cdots \beta_{nn} \end{vmatrix}}{\begin{vmatrix} \beta_{11} & \beta_{12} \cdots \beta_{1n} \\ \beta_{21} & \beta_{22} \cdots \beta_{2n} \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot \\ \beta_{n1} & \beta_{n2} \cdots \beta_{nn} \end{vmatrix}} = \frac{\frac{15}{4} N}{\beta_{11} + \beta_{12}\frac{D_2}{D} + \beta_{13}\frac{D_3}{D} + \ldots}$$

where

$$D = \lim_{n\to\infty} \begin{vmatrix} \beta_{22} \cdots \beta_{2n} \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \beta_{n2} \cdots \beta_{nn} \end{vmatrix}, \qquad D_i = \lim_{n\to\infty} \begin{vmatrix} \beta_{21} \cdots \beta_{2,i-1} & \beta_{2,i+1} \cdots \beta_{2n} \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \beta_{n1} \cdots \beta_{n,i-1} & \beta_{n,i+1} \cdots \beta_{nn} \end{vmatrix}$$

A sufficiently good accuracy is obtained if we take only the first terms of the series in the denominator of the above equation; this corresponds to the first-order Chapman-Enskog approximation for expansion in powers of a small parameter, and we obtain $a_1 = 15N/4\beta_{11}$, so that $\varkappa = (75T/8m)(N^2/\beta_{11})$.

Let us find β_{11}:

$$\sqrt{\frac{m}{2T}}\,\beta_{11} = \int f_1^{(0)} f_2^{(0)} \left[\mathbf{u}_1\left(u_1^2 - \frac{5}{2}\right) + \mathbf{u}_2\left(u_2^2 - \frac{5}{2}\right) - \mathbf{u}_1'\left(u_1'^2 - \frac{5}{2}\right)\right.$$

$$\left. - \mathbf{u}_2'\left(u_2'^2 - \frac{5}{2}\right)\right]\mathbf{u}_1\left(u_1^2 - \frac{5}{2}\right)|\mathbf{u}_1 - \mathbf{u}_2|\, d\sigma\, d\mathbf{u}_1\, d\mathbf{u}_2$$

It can be seen that the integral is not changed upon exchanging the places of the particles 1 and 2. Therefore, we can make the replacement $\mathbf{u}_1 \to \mathbf{u}_2$, $\mathbf{u}_2 \to \mathbf{u}_1$ in the integrand. Moreover, since the laws of mechanics are reversible with respect to the time reflection $t \to -t$, the integral is not changed if we make the replacement $\mathbf{u} \to -\mathbf{u}'$, $\mathbf{u}' \to -\mathbf{u}$. Hence, we can express the solution in a symmetric form, replacing $\mathbf{u}_1\left(u_1^2 - \frac{5}{2}\right)$ in the integrand by

$$\frac{1}{4}\left[\mathbf{u}_1\left(u_1^2-\frac{5}{2}\right)+\mathbf{u}_2\left(u_2^2-\frac{5}{2}\right)-\mathbf{u}_1'\left(u_1'^2-\frac{5}{2}\right)-\mathbf{u}_2'\left(u_2'^2-\frac{5}{2}\right)\right],$$

We obtain

$$\sqrt{\frac{m}{2T}}\,\beta_{11}=\int f_1^{(0)}f_2^{(0)}\frac{1}{4}\left[\mathbf{u}_1\left(u_1^2-\frac{5}{2}\right)+\mathbf{u}_2\left(u_2^2-\frac{5}{2}\right)\right.$$
$$\left.-\,\mathbf{u}_1'\left(u_1'^2-\frac{5}{2}\right)-\mathbf{u}_2'\left(u_2'^2-\frac{5}{2}\right)\right]^2|\mathbf{u}_1-\mathbf{u}_2|\,d\sigma\,d\mathbf{u}_1\,d\mathbf{u}_2$$

Let us make use of the laws of conservation of momentum and energy assuming that the colliding particles are identical. Denote by $\mathbf{w}$ and $\mathbf{w}'$ the dimensionless relative velocities of the particles before and after the collision, and by $\mathbf{G}$ the dimensionless velocity of their centre of inertia. Conservation of momentum yields $\mathbf{G}=\mathbf{G}'$ and conservation of energy in elastic scattering of particles yields $w=w'$. Hence we obtain $\mathbf{u}_1=\mathbf{G}+\mathbf{w}/2$, $\mathbf{u}_2=\mathbf{G}-\mathbf{w}/2$, $\mathbf{u}_1'=\mathbf{G}+\mathbf{w}'/2$, and $\mathbf{u}_2'=\mathbf{G}-\mathbf{w}'/2$, so that

$$\left[\mathbf{u}_1\left(u_1^2-\frac{5}{2}\right)+\left(u_2^2-\frac{5}{2}\right)\mathbf{u}_2-\mathbf{u}_1'\left(u_1'^2-\frac{5}{2}\right)-\mathbf{u}_2'\left(u_2'^2-\frac{5}{2}\right)\right]$$
$$=[\mathbf{w}(\mathbf{w}\mathbf{G})-\mathbf{w}'(\mathbf{w}'\mathbf{G})].$$

Hence we find

$$\frac{1}{N^2}\sqrt{\frac{m}{2T}}\,\beta_{11}=\int\exp\left(-2G^2-w^2/2\right)[\mathbf{w}(\mathbf{w}\mathbf{G})-\mathbf{w}'(\mathbf{w}'\mathbf{G})]^2w\,d\sigma\,\frac{d\mathbf{w}\,d\mathbf{G}}{4\pi^3}$$

Denoting by θ the scattering angle, we have $\mathbf{w}\mathbf{w}'=w^2\cos\theta$. Let the polar axis be directed along $\mathbf{G}$. The position of the vector $\mathbf{w}$ is described by the polar angle ϑ, and we have $\mathbf{w}\mathbf{G}=wG\cos\vartheta$, $\mathbf{w}'\mathbf{G}=wG(\cos\vartheta\cos\theta+\sin\vartheta\sin\theta\cos\Phi)$ and $d\mathbf{w}=w^2\,dw\,d(\cos\vartheta)\,d\Phi$. We obtain

$$\beta_{11}=12\sqrt{\frac{T}{\pi m}}\,N^2\bar{\sigma}_2$$

where

$$\bar{\sigma}_2=\frac{1}{48}\int_0^\infty\exp\left(-w^2/2\right)w^7\sigma_2(w)\,dw,\qquad \sigma_2=\int(1-\cos^2\theta)\,d\sigma \qquad (1.34)$$

Hence heat conductivity is

$$\varkappa = \frac{25\sqrt{\pi T}}{32\sqrt{m}\,\sigma_2} \tag{1.35}$$

If scattering of the atoms can be described in the framework of the rigid sphere model, we obtain $d\sigma = (1/2)\pi r_0^2 d\,(\cos\theta)$, where r_0 is the sphere radius. Then $\bar{\sigma}_2 = \sigma_2 = (2/3)\pi r_0^2$ and

$$\varkappa = \frac{75\sqrt{T}}{64\sqrt{\pi m}\,r_0^2} \tag{1.36}$$

Equation (1.35) shows that heat conductivity $\varkappa$ is independent of the number density of the gas particles. Indeed, the number of carriers varies proportionally to the number density of particles, and the mean free path of the carriers is inversely proportional to the number density. Therefore, these two effects fully compensate for each other in binary collisions of gas particles.

Problem 1.14. Calculate the heat conductivity of a monoatomic gas if the frequency of collisions of gas atoms is independent of the relative velocity of collisions.

The kinetic equation (1.31) describes heat transfer in our case. We shall write this equation as

$$f_1^{(0)}\left(\frac{mv_1^2}{2T} - \frac{5}{2}\right) v_{1x} \frac{\partial \ln T}{\partial x} = \int (f_1' f_2' - f_1 f_2)\,|\mathbf{v}_1 - \mathbf{v}_2|\,d\sigma\,d\mathbf{v}_2$$

where $f_1^{(0)}$ is the Maxwellian velocity distribution function, f_1 and f_2 are the exact distribution functions taking into account heat transfer, and the gas temperature T varies along the x-axis. Multiply this equation by $v_{1x}v_1^2$ and integrate the result with respect to the atoms' velocities. Using eq. (1.28) for the Maxwellian velocity distribution, we obtain the following expression for the left-hand side of the equation:

$$\int v_{1x} v_1^2 d\mathbf{v}_1 f_1^{(0)}\left(\frac{mv_1^2}{2} - \frac{5}{2}\right) v_{1x} \frac{\partial \ln T}{\partial x} = \frac{5T}{m^2}\frac{\partial T}{\partial x} N$$

Here N is the density of the gas atoms. In the right-hand side we shall make the replacement $\mathbf{v}_{1,2} \to \mathbf{v}'_{1,2}$, $\mathbf{v}'_{1,2} \to \mathbf{v}_{1,2}$ and using the principle of detailed balancing for the differential cross section of elastic scattering, we obtain

$$\int (v'_{1x} v_1'^2 - v_{1x} v_1^2) f_1 f_2\,|\mathbf{v}_1 - \mathbf{v}_2|\,d\sigma\,d\mathbf{v}_1\,d\mathbf{v}_2$$

Since the colliding particles are of the same species, the differential cross section $d\sigma$ of elastic scattering is invariant under replacement of the particle's number ($\mathbf{v}_1 \rightleftarrows \mathbf{v}_2$). After this replacement and symmetrization of the right-hand side, we obtain

$$\frac{5T^2}{m^2}\frac{\partial T}{\partial x}N = \frac{1}{2}\int (v'_{1x}v'^2_1 + v'_{2x}v^2_2 - v_{1x}v^2_1 - v_{2x}v^2_2)\,|\mathbf{v}_1 - \mathbf{v}_2|\,d\sigma\,d\mathbf{v}_1\,d\mathbf{v}_2$$

To evaluate the integral over the scattering angles

$$\int (v'_{1x}v'^2_1 + v'_{2x}v'^2_2 - v_{1x}v^2_1 - v_{2x}v^2_2)\,d\sigma$$

we shall make use of the relative velocity of collision $\mathbf{g} = \mathbf{v}_1 - \mathbf{v}_2$ and the velocity of the centre of inertia $\mathbf{v}_c = (\mathbf{v}_1 + \mathbf{v}_2)/2 = (\mathbf{v}'_1 + \mathbf{v}'_2)/2$. The relative velocity of atoms after collision is

$$\mathbf{g}' = \mathbf{g}\cos\theta + \mathbf{n}g\sin\theta$$

where θ is the scattering angle and $\mathbf{n}$ is the unit vector perpendicular to $\mathbf{g}$. Hence we obtain

$$\int (\mathbf{v}'_1 v'^2_1 + \mathbf{v}'_2 v'^2_2 - \mathbf{v}_1 v'^2_1 - \mathbf{v}_2 v^2_2)\,d\sigma = \int [\mathbf{g}'(\mathbf{v}_c\mathbf{g}') - \mathbf{g}(\mathbf{v}_c\mathbf{g})]\,d\sigma$$
$$= -\mathbf{g}(\mathbf{v}_c\mathbf{g})\int (1 - \cos^2\theta)\,d\sigma + \mathbf{g}\int \cos\theta(\mathbf{v}_c\mathbf{n})g\sin\theta\,d\sigma$$
$$+ \mathbf{g}\int \mathbf{n}\sin\theta\,(\mathbf{v}_c\mathbf{g})\cos\cancel{\theta\,d\sigma} + g\int \mathbf{n}\sin\theta\,(\mathbf{v}_c\mathbf{n})\cdot g\sin\theta\,d\sigma$$

Since the vector $\mathbf{n}$ can have any direction in the plane perpendicular to $\mathbf{g}$ with the same probability, the averaging over the direction of $\mathbf{n}$ turns the second and third terms zero. The vector $\mathbf{v}_c$ in the last term can be represented as a sum of two components — one parallel to $\mathbf{g}$ and another perpendicular to it (see Fig. 1.1).

The first term is zero, since the vector $\mathbf{n}$ is, by definition, perpendicular to $\mathbf{g}$. Thus, the second term determines the direction of the sought vector. Taking this into account and averaging over the components of the unit vector, we finally obtain

$$\int (\mathbf{v}'_1 v'^2_1 + \mathbf{v}'_2 v'^2_2 - \mathbf{v}_1 v^2_1 - \mathbf{v}_2 v^2_2)\,d\sigma$$
$$= -\mathbf{g}(\mathbf{v}_c\mathbf{g})\sigma^{(2)}(g) - \frac{\mathbf{n}_0}{2}\mathbf{v}_c g^2\sigma^{(2)}(g) =$$
$$= -\frac{1}{2}[\mathbf{g}(\mathbf{v}_c\mathbf{g}) + \mathbf{v}_c g^2]\sigma^{(2)}(g)$$

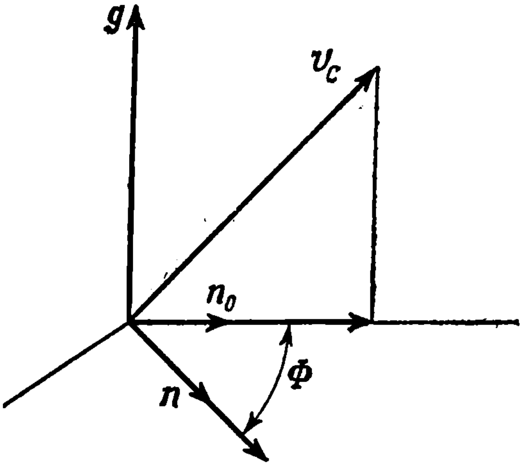

Fig. 1.1. Averaging: $\overline{g^2\mathbf{n}(\mathbf{v}_c\mathbf{n})} = g^2\mathbf{v}_c\mathbf{n}_0 \times \overline{\cos^2\Phi} = \frac{1}{2}g^2\mathbf{v}_c\mathbf{n}_0 = \frac{1}{2}\mathbf{v}_c g^2 - \frac{1}{2}\mathbf{g}(\mathbf{v}_c\mathbf{g})$; Φ is the azimuthal angle in the direction perpendicular to $\mathbf{g}$.

Here $\sigma^{(2)}(g) = \int (1 - \cos^2\theta)\, d\sigma$ is the scattering cross section of the atoms averaged over the angles, and $\mathbf{n}_0$ is the unit vector perpendicular to $\mathbf{g}$ and lying in the plane of the vectors $\mathbf{v}_c$ and $\mathbf{g}$. Then we obtain for the sought equation

$$\frac{5T}{m^2}\frac{\partial T}{\partial x}N = -\frac{1}{4}\int [g_x(\mathbf{v}_c\mathbf{g}) + (\mathbf{v}_c)_x g^2]\, g\sigma^{(2)}(g) f_1 f_2\, d\mathbf{v}_1\, d\mathbf{v}_2$$

A condition of the problem is that the collision frequency $k^{(2)} = g\sigma^{(2)}(g)$ is independent of the velocity of the particles. Express the relative velocity and the velocity of the centre of inertia in terms of the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of the colliding particles. Since the above integrand is symmetric with respect to the number of a particle, we can write

$$\frac{5T}{m^2}\frac{\partial T}{\partial x}N = -\frac{k^{(2)}}{2}\int [v_{1x}v_1^2 - v_{1x}(\mathbf{v}_1\mathbf{v}_2)] f_1 f_2\, d\mathbf{v}_1\, d\mathbf{v}_2 \qquad (1.37)$$

The second term of the integrand has a higher order of smallness than the first one. Indeed, in the zero approximation the distribution functions are spherically symmetric, so that the second term should take into account the terms of the second order of smallness. Neglecting it and using the definition of the heat flux $\mathbf{q} = \int \mathbf{v}_1 (m v_1^2/2) f_1\, d\mathbf{v}_1$, we obtain from the above equation

$$\frac{5T}{m^2}\frac{\partial T}{\partial x}N = -\frac{k^{(2)}}{m}Nq_x$$

Comparing this equation with the definition of heat conductivity ($\mathbf{q} = -\varkappa\nabla T$), we find

$$\varkappa = 5T/mk^{(2)} \qquad (1.38)$$

This is the exact result, since in its derivation we did not make additional assumptions on the velocity dependence of the correction term for the distribution function. However, the Chapman-Enskog approximation (1.35) yields the same result for the condition of this problem ($k^{(2)} = \text{const}$).

Problem 1.15. Find the relationship between the fluxes of particles and gradients of the appropriate parameters for a mixture of two gases.

Define the molar concentration c of particles of a given species as the ratio of the number of particles of this species to the total number of particles ($c = N_1/N$, where N_1 is the number density of the first species and $N = N_1 + N_2$ is the total density). Hence, we obtain for the first term of the left-hand side of the kinetic equation (1.30)

$$\mathbf{v}_1 \nabla f_1^0 = \mathbf{v}_1 f_1^{(0)} \left[\nabla \ln p_1 + \left(\frac{m v_1^2}{2T} - \frac{5}{2} \right) \nabla \ln T \right]$$

$$= \mathbf{v}_1 f_1^{(0)} \left[\nabla \ln p + \nabla \ln c + \left(\frac{m v_1^2}{2T} - \frac{5}{2} \right) \nabla \ln T \right]$$

since $\nabla \ln p_1 = \nabla \ln NcT = \nabla \ln p + \nabla \ln c$.

Find the force acting on a gas volume and maintaining its equilibrium. Neglecting the gas viscosity, we shall make use of the Euler equation (1.18) and write

$$M \frac{d\langle \mathbf{v} \rangle}{dt} = 0 = - M \frac{\nabla p}{\rho} + \frac{M}{(m)_c} [c\mathbf{F}_1 + (1 - c)\mathbf{F}_2] + M\mathbf{f}$$

where angle brackets denote averaging over velocities of the particles. Here M is the mass of the gas under consideration, $\rho = N_1 m_1 + N_2 m_2 = N m_c$ is the gas density (mass in the unit gas volume), $\mathbf{F}_1$ and $\mathbf{F}_2$ are the external forces acting on each species, $m_c = m_1 c + m_2(1 - c)$, so that M/m_c is the number of particles in the gas with the mass M, $\mathbf{f}$ is the force acting on the unit mass of the gas and needed to maintain equilibrium in the system. Hence we obtain for the second term of the left-hand side of the kinetic equation (1.30)

$$\frac{\mathbf{F}}{m_1} \frac{\partial f_1^{(0)}}{\partial \mathbf{v}_1} = \left(\frac{\mathbf{F}_1}{m_1} + \mathbf{f} \right) \frac{\partial f_1^{(0)}}{\partial \mathbf{v}_1}$$

$$= - \frac{m_1 \mathbf{v}_1}{T} f_1^{(0)} \left[\frac{\mathbf{F}_1}{m_1} + \frac{\nabla p}{N m_c} - \frac{c\mathbf{F}_1 + (1 - c)\mathbf{F}_2}{m_c} \right]$$

and the left-hand side of eq. (1.30) is

$$f_1^{(0)} \left[\frac{\mathbf{v}_1 \mathbf{d}_1}{c} + \left(\frac{m v_1^2}{2T} - \frac{5}{2} \right) \mathbf{v}_1 \nabla \ln T \right]$$

where

$$\mathbf{d}_1 = \nabla c + \frac{c(1 - c)(m_2 - m_1)}{m_c} \nabla \ln p + \frac{c(1 - c)}{T m_c} (m_1 \mathbf{F}_2 - m_2 \mathbf{F}_1) \quad (1.39)$$

We shall solve the kinetic equation here as it was done for the case of heat conduction. We obtain the following system of integro-differential equations for the functions $\theta_i = f_i^{(1)}/f_i^{(0)}$:

$$\frac{f_1^{(0)}}{c} \mathbf{v}_i \mathbf{d}_i + f_i^{(0)} \mathbf{v}_i \left(\frac{m v_i^2}{2T} - \frac{5}{2} \right) \nabla \ln T$$

$$= \sum_{k=1,2} \int f_i^{(0)} f_k^{(0)} (\theta_i + \theta_k - \theta_i' - \theta_k') \, |\mathbf{v}_i - \mathbf{v}_k| \, d\sigma_{ik} \, d\mathbf{v}_k \quad (1.40)$$

Here $i = 1, 2$, the vector $\mathbf{d}_1$ is given by eq. (1.39), and $\mathbf{d}_2 = -\mathbf{d}_1$. The solution of eq. (1.40) can be written as $\theta_i = \mathbf{B}_i\mathbf{d} + \mathbf{L}_i\nabla \ln T(\mathbf{d} = \mathbf{d}_1)$ and, since the gradients are small, the vectors $\mathbf{B}_i$ and $\mathbf{L}_i$ are parallel to the velocity $\mathbf{v}_i$ of a given species. When the solution is written as

$$\theta_i = B_i(v_i)\mathbf{v}_i\mathbf{d} + L_i(v_i)\mathbf{v}_i\nabla \ln T \tag{1.41}$$

conditions (1.29a and c) are automatically satisfied, since the distribution function $f_i^{(1)}$ is odd with respect to $\mathbf{v}_i$.

Let us make use of the coordinate system in which there is no mass flow of the particles ($\rho_1\mathbf{v}_{10} + \rho_2\mathbf{v}_{20} = 0$). Then eq. (1.29b) is replaced by the condition

$$m_1\int f_1^{(0)}\theta_1\mathbf{v}_1\,d\mathbf{v}_1 + m_2\int f_2^{(0)}\theta_2\mathbf{v}_2\,d\mathbf{v}_2 = 0 \tag{1.42}$$

Expand B_i and L_i in eq. (1.41) in the Sonin polynomials:

$$B_i = \sum_{m=0}^{\infty} b_i^m S_{3/2}^m(u_i^2), \qquad L_i = \sum_{m=0}^{\infty} l_i^m S_{3/2}^m(u_i^2)$$

The basis $S_{3/2}^m$ is convenient here since it reduces condition (1.42) to the simple relationships

$$cb_1^{(0)} + (1-c)b_2^{(0)} = 0$$
$$cl_1^{(0)} + (1-c)l_2^{(0)} = 0$$

Calculate the mean velocity of a given species in the coordinate system where there is no mass flow. We have

$$\begin{aligned}\langle\mathbf{v}_i\rangle &= \frac{1}{N_i}\int f_i^{(0)}[B_i\mathbf{d} + L_i\nabla \ln T]\mathbf{v}_i\,d\mathbf{v}_i \\ &= \frac{1}{N_i}\int f_i^{(0)}\mathbf{v}_i\,d\mathbf{v}_i \sum_m [\mathbf{v}_i\mathbf{d}b_i^m S_{3/2}^m(u_i^2) + \mathbf{v}_i\nabla \ln T l_i^m S_{3/2}^m(u_i^2)] \\ &= \frac{T}{m_i}[b_i^{(0)}\mathbf{d} + l_i^{(0)}\nabla \ln T]\end{aligned}$$

Hence we can find the diffusion flow, that is, the flow of the particles of a given species in the coordinate system where there is no flow of gas as a whole (the mass flow). We obtain

$$\begin{aligned}\mathbf{j} &= N_1\langle\mathbf{v}_1\rangle - c(N_1\langle\mathbf{v}_1\rangle + N_2\langle\mathbf{v}_2\rangle) = Nc(1-c)(\langle\mathbf{v}_1\rangle - \langle\mathbf{v}_2\rangle) \\ &= Nc(1-c)\left[\frac{T}{m_1}b_1^{(0)}\mathbf{d} + \frac{T}{m_1}l_1^{(0)}\nabla \ln T - \frac{T}{m_2}b_2^{(0)}\mathbf{d} - \frac{T}{m_2}l_2^{(0)}\nabla \ln T\right] \\ &= NcT\frac{m_c}{m_1m_2}[b_1^{(0)}\mathbf{d} + l_2^{(0)}\nabla \ln T]\end{aligned}$$

In the absence of gradients of temperature and pressure we have only the diffusion flux, which is $\mathbf{j} = -D_{12}N\nabla c$, by definition. Since in this case $\mathbf{d} = \nabla c$, we have

$$D_{12} = -\frac{Tm_c c}{m_1 m_2} b_1^{(0)}$$

The proportionality factor between the flux $\mathbf{j}$ and the gradient $\nabla \ln T$ is known as the thermal diffusion coefficient D_T. Thus, the flux of particles of a given species can be expressed as

$$\mathbf{j} = -D_{12}N\mathbf{d} - D_T\nabla \ln T \tag{1.43}$$

We see that the diffusion coefficient and the thermal diffusion coefficient are expressed in terms of the first coefficients of the expansion of the distribution function in Sonin polynomials. Thus the coefficients of thermal diffusion and diffusion can be found from the solution of the kinetic equation.

Problem 1.16. Find the diffusion coefficient in the Chapman-Enskog approximation.

Let us trace the motion of a group of test particles whose number density is small compared with the number density of the gas. Hence, the collisions between these particles can be ignored, their concentration is small and the diffusion flux is

$$\mathbf{j} = -\mathscr{D}\nabla N_1 \tag{1.44}$$

where $\mathscr{D}$ is the diffusion coefficient and N_1 is the number density of the test particles. The kinetic equation (1.30) for the distribution function of the test particles taking into account their diffusion is

$$\mathbf{v}_1 f_1^{(0)} \frac{\nabla N_1}{N_1} = \int (f_1' f_2' - f_1 f_2)|\mathbf{v}_1 - \mathbf{v}_2|\, d\sigma\, d\mathbf{v}_2$$

where $f_1^{(0)}$ is the Maxwellian distribution function for the test particles, f_1 is the distribution function of the test particles taking into account diffusion and f_2 is the Maxwellian distribution function of the gas particles.

Let the density gradient be directed along the x-axis. Multiply the above equation by the momentum Mv_{1x} of the test particle in this direction and integrate the result over the velocities. The left-hand side of the resulting equation is $T(\partial N_1/\partial x)$ and the whole equation can be written as

$$T\frac{\partial N_1}{\partial x} = -\mu \int f_1 f_2 g_x g \sigma^*(g)\, d\mathbf{v}_1\, d\mathbf{v}_2 \tag{1.45}$$

where $\mathbf{g} = \mathbf{v}_1 - \mathbf{v}_2$ is the relative velocity of collision, $\sigma^*(g)$ is the diffusion cross section of the collision between the test particle and a gas particle, and μ is the reduced mass of the test particle and the gas particle.

The distribution function for the test particles in the Chapman-Enskog approximation can be written as

$$f(\mathbf{v}_1) = f^{(0)}(v_1)\left[1 - v_{1x}\frac{\partial \ln N_1}{\partial x}h(v_1)\right]$$

Substituting this equation into the right-hand side of eq. (1.45) and assuming that in the first approximation $h(v_1) = \text{const}$, we obtain

$$T\frac{\partial N_1}{\partial x} = \frac{\partial N_1}{\partial x}\frac{\mu m_2}{m_1 + m_2}\langle g_x^2 g\sigma^*(g)\rangle h$$

We have used here the relationship $\mathbf{v}_1 = \mathbf{v}_c + \dfrac{m_2}{m_1 + m_2}\mathbf{g}$, where $\mathbf{v}_c$ is the velocity of the centre of inertia, and m_1 and m_2 are the masses of the test particle and the gas particle. The angle brackets denote averaging over the Maxwellian distribution and the fact that $\langle \mathbf{v}_c\rangle = 0$ has been taken into account. Hence we obtain

$$h = \frac{3Tm_1}{\mu^2 N\langle g^3\sigma^*\rangle}$$

The flow of the test particles in this case is

$$j_x = \int v_{1x}f(v_1)\,d\mathbf{v}_1 = -\frac{\partial \ln N_1}{\partial x}h\int v_{1x}^2 f_1^{(0)}\,d\mathbf{v}_1 = -\frac{hT}{m_1}\frac{\partial N_1}{\partial x}$$

Substituting into this equation the above expression for h and comparing the result with the definition of the diffusion coefficient, we find the diffusion coefficient in the first Chapman-Enskog approximation:

$$\mathscr{D}_{12} = \frac{3T^2}{\mu^2\langle g^3\sigma^*(g)\rangle N} = \frac{3\sqrt{\pi T}}{8N\sqrt{2\mu}\bar{\sigma}} \tag{1.46}$$

$$\bar{\sigma} = \frac{1}{8}\int_0^\infty \exp\left(-\frac{w^2}{2}\right)w^5\sigma^*(w)\,dw$$

Here $\sigma^*(w) = \int(1 - \cos\theta)\,d\sigma$ is the diffusion cross section of scattering for collisions of particles of different species, $w = g/\sqrt{2T/\mu}$ is the reduced relative velocity of collision, and N is the number density of the gas particles.

In the rigid sphere model $\bar{\sigma} = \sigma^* = \pi r_0^2$, so that

$$\mathscr{D}_{12} = \frac{3}{8Nr_0^2}\left(\frac{T}{2\pi\mu}\right)^{1/2}$$

Problem 1.17. Derive the diffusion coefficient if the collision frequency is independent of the collision velocity.

As before, we assume that the number density of the test particles is small compared with the number density of gas particles. Then the flux of the test particles is

$$j_x = -\mathscr{D}\frac{\partial N_1}{\partial x} = N_1\langle v_{1x}\rangle$$

where x is the axis along which the number density of the test particles varies slowly. We shall make use of eq. (1.26a) derived in Problem 1.11 for the mean velocity of particles under the same conditions:

$$\langle v_{1x}\rangle = -\frac{m_1(\langle v_{1x}^2\rangle - \langle v_{1x}\rangle^2)}{\mu\nu}\frac{\partial \ln N_1}{\partial x}$$

Here $\nu = Ng\sigma^*(g)$ is the frequency of collisions of the test particle with the gas particles, N is the number density of the gas particles, and $\sigma^*(g)$ is the diffusion cross section for collisions of the particles. In this case the collision frequency ν is independent of the collision velocity.

The above equations yield the coefficient of diffusion of the test particles in the gas:

$$\mathscr{D} = \frac{m_1(\langle v_x^2\rangle - \langle v_x\rangle^2)}{\mu\nu} \tag{1.47}$$

This relationship has been derived without additional assumptions and is valid in the general case, including the case when the distribution function is spherically asymmetric (motion of charged particles in external fields). If the unperturbed distribution function is close to the Maxwellian distribution (motion of the test particles in the gas under the condition close to equilibrium), eq. (1.47) is transformed into

$$D = \frac{T}{\mu\nu}$$

This equation is identical to that obtained in the first Chapman-Enskog approximation if the collision frequency in eq. (1.46) is assumed to be independent of the relative collision velocity.

Problem 1.18. Determine the viscosity of the monoatomic gas if the collisions between its atoms are elastic.

In the case of directed motion of a gas without temperature and pressure gradients the left-hand side of the kinetic equation (1.30) can be written as

$$\mathbf{v}\nabla f^{(0)} = v_i\frac{\partial f^{(0)}}{\partial x_i} = v_i\,\frac{m(v_k - v_{0k})}{T}\,\frac{\partial v_{0k}}{\partial x_i}f^{(0)}$$

$$= \frac{m(v_i - v_{0i})(v_k - v_{0k})}{T}\,\frac{\partial v_{0k}}{\partial x_i}f^{(0)} + \frac{m(v_k - v_{0k})}{T}f^{(0)}v_{0i}\frac{\partial v_{0k}}{\partial x_i}$$

Here the subscripts i and k denote the respective projections of vectors and if a subscript is repeated twice, summation is implied. Equation (1.28) for the distribution function $f^{(0)}$ has been used here. Let us make use of the Euler equation yielding $v_{0i}(\partial v_{0k}/\partial x_i) = -(1/\rho)(\partial p/\partial x_k)$. Since the pressure is spatially uniform, the second term in the above equation is zero. Let us write the left-hand side of the kinetic equation in a more symmetric form

$$v_i\frac{\partial f^{(0)}}{\partial x_i} = \left[\left(u_iu_k - \frac{u^2}{3}\delta_{ik}\right)\left(\frac{\partial v_{0i}}{\partial x_k} + \frac{\partial v_{0k}}{\partial x_i}\right) + \frac{2}{3}u^2\,\mathrm{div}\,\mathbf{v}_0\right]f^{(0)}$$

where $u_i = (v_i - v_{0i})\sqrt{m/2T}$. The continuity equation yields $\mathrm{div}\,N\mathbf{v}_0 = N\mathrm{div}\,\mathbf{v}_0 + \mathbf{v}_0\nabla N = 0$ and $\mathrm{div}\,\mathbf{v}_0 = 0$, since the pressure and temperature gradients are zero according to the conditions of this problem. Hence we obtain the kinetic equation in the following form:

$$\left(u_iu_k - \frac{u^2}{3}\delta_{ik}\right)\left(\frac{\partial v_{0k}}{\partial x_i} + \frac{\partial v_{0i}}{\partial x_k}\right)f^{(0)}$$

$$= \int f^{(0)}f_1^{(0)}(\theta' + \theta_1' - \theta - \theta_1)\,|\mathbf{v} - \mathbf{v}_1|\,d\sigma\,d\mathbf{v}_1 \qquad (1.48)$$

We shall find the solution of this equation in the form

$$\theta = -C_{ik}\left(\frac{\partial v_{0k}}{\partial x_i} + \frac{\partial v_{0i}}{\partial x_k}\right)$$

where the tensor C_{ik} is constructed of the vectors v_i. The tensor C_{ik} is symmetric, so that it is a combination of the tensors v_iv_k and δ_{ik}. However, to satisfy conditions (1.29a and c) the tensor C_{ik} should be proportional to the following combination of these tensors:

$$C_{ik} = C(u)\left(u_iu_k - \frac{u^2}{3}\delta_{ik}\right), \quad \text{where} \quad \mathbf{u} = \sqrt{\frac{m}{2T}}(\mathbf{v} - \mathbf{v}_0)$$

Hence we obtain

$$\theta = -C(u)\left(u_iu_k - \frac{u^2}{3}\delta_{ik}\right)\left(\frac{\partial v_{0i}}{\partial x_k} + \frac{\partial v_{0k}}{\partial x_i}\right) \qquad (1.49)$$

Expand $C(u)$ in the Sonin polynomials:

$$C(u) = \sum c_m S^m_{5/2}(u^2) \tag{1.50}$$

The basis $S^m_{5/2}$ is convenient here because the viscosity is expressed in terms of the first coefficient of the expansion in this case. Let us demonstrate this. The pressure tensor $P_{ik} = \langle m(v_i - v_{0i})(v_k - v_{0k})\rangle$ can be expressed as the following sum (div $\mathbf{v}_0 = 0$):

$$P_{ik} = p\delta_{ik} - \eta\left(\frac{\partial v_{0i}}{\partial x_k} + \frac{\partial v_{0k}}{\partial x_i}\right)$$

where p is the gas pressure, and η is the gas viscosity. On the other hand, if $i \neq k$, we have

$$P_{ik} = m\int (v_i - v_{0i})(v_k - v_{0k}) f \, d\mathbf{v}$$

$$= -2T\int u_i u_k f^{(0)} \, d\mathbf{v}\, C(u)\left(u_l u_j - \frac{u^2}{3}\delta_{lj}\right)\left(\frac{\partial v_{0l}}{\partial x_j} + \frac{\partial v_{0j}}{\partial x_l}\right)$$

$$= -4T\left(\frac{\partial v_{0i}}{\partial x_k} + \frac{\partial v_{0k}}{\partial x_i}\right)\int u_i^2 u_k^2 C(u) f^{(0)} \, d\mathbf{v}$$

since in the sum over the subscripts l and j only the terms for which $l = i$, $j = k$ or $l = k$, $j = i$ differ from zero. For other terms integration over the angles yields zero. Using this equation, we find the viscosity

$$\eta = 4T\int u_i^2 u_k^2 C(u) f^{(0)} \, d\mathbf{v} = \frac{4T}{15}\int u^4 C(u) f^{(0)} \, d\mathbf{v}$$

Using the expansion (1.50) for $C(u)$, we obtain

$$\eta = \frac{4T}{15}\frac{2N}{\sqrt{\pi}}\sum_m c_m \int_0^\infty z^{5/2} \exp(-z) S^m_{5/2}(z) \, dz = TNc_0 \tag{1.51}$$

that is, the viscosity is expressed in terms of the coefficient of the zero term in the expansion of the function $C(u)$ in the Sonin polynomials.

We shall solve the kinetic equation (1.48) for viscosity as it was done above in the case of heat conduction. Multiply eq. (1.48) from the left by the tensor $S^m_{5/2}(u^2)u_{jl}$ (where $u_{jl} = u_j u_l - (u^2/3)\delta_{jl}$ and $j \neq l$) and integrate the result over velocities. The following notation will be used here:

$$\beta_q = \int u_{ik} u_{ik} f^{(0)} S^q_{5/2}(u^2) \, d\mathbf{u} = N\int u_i^2 u_k^2 S^q_{5/2}(u^2) \exp(-u^2) \frac{4}{\sqrt{\pi}} u^2 \, du = \frac{N}{4}\delta_{q0}$$

and

$$\gamma_{qp} = \int f^{(0)} f_1^{(0)} [S^p_{5/2}(u^2) u_{ik} + S^p_{5/2}(u_1^2) u_{1\,ik} - S^p_{5/2}(u'^2) u'_{ik} - S^p_{5/2}(u_1'^2) u'_{1\,ik}] \, |\mathbf{v} - \mathbf{v}_1| \, d\sigma \, S^q_{5/2}(u^2) u_{ik} \, d\mathbf{u} \, d\mathbf{u}_1$$

There is no summation over the subscripts i and k here ($i \neq k$).

We obtain the following system of equations for the coefficients of expansion (1.50):

$$\beta_q = \sum_p c_q \gamma_{qp}$$

Solving this system of equation we obtain the sought coefficient c_0:

$$c_0 = \lim_{n \to \infty} \frac{\beta_0}{\gamma_{00} + \dfrac{1}{D_n} \sum\limits_{s=1}^{n} D_{ns}}.$$

where the determinant $D_n = \begin{vmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{vmatrix}$ and the determinant D_{ns} is obtained from the determinant

$$\begin{vmatrix} \gamma_{00} & \gamma_{01} & \cdots & \gamma_{0n} \\ \gamma_{10} & \gamma_{11} & \cdots & \gamma_{1n} \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \gamma_{n0} & \gamma_{n1} & \cdots & \gamma_{nn} \end{vmatrix}$$

by deleting the zero row and the sth column. According to the Chapman-Enskog approximation, the first term of the expansion in the denominator gives a sufficient accuracy of the solution. Therefore, in the first Chapman-Enskog approximation we obtain

$$c_0 = \frac{\beta_0}{\gamma_{00}} = \frac{N}{4\gamma_{00}}$$

Let us find the quantity

$$\gamma_{00} = \int f^{(0)} f_1^{(0)} (u_{ik} + u_{1\,ik} - u'_{ik} - u'_{1\,ik}) u_{ik} \, |\mathbf{v} - \mathbf{v}_1| \, d\sigma \, d\mathbf{u}_1 \, d\mathbf{u}$$

Since the integrand is symmetric with respect to the replacements $\mathbf{u}, \mathbf{u}' \rightleftarrows \mathbf{u}_1, \mathbf{u}_1'$ and $\mathbf{u} \rightleftarrows \mathbf{u}'$, $\mathbf{u}_1 \rightleftarrows \mathbf{u}_1'$ we can write this quantity as

$$\gamma_{00} = \frac{1}{4} \int f^{(0)} f_1^{(0)} (u_{ik} + u_{1\,ik} - u'_{ik} - u'_{1\,ik})^2 \, |\mathbf{v} - \mathbf{v}_1| \, d\sigma \, d\mathbf{u} \, d\mathbf{u}_1$$

Let us make use of the coordinate system in which the polar z-axis corresponds to the subscript i and the x-axis corresponds to the subscript k. The reduced relative velocity is $\mathbf{w} = \mathbf{u} - \mathbf{u}_1$ and the velocity of the centre of inertia is $\mathbf{G} = (\mathbf{u} + \mathbf{u}_1)/2$. Since $d\mathbf{u}_1\, d\mathbf{u} = d\mathbf{w}\, d\mathbf{G}$ (because $\mathbf{u} = \mathbf{G} + \mathbf{w}/2$, $\mathbf{u}_1 = \mathbf{G} - \mathbf{w}/2$), we obtain

$$\gamma_{00} = \frac{1}{4}\int f^{(0)} f_1^{(0)} \Bigg[\left(G_z + \frac{w_z}{2}\right)\left(G_x + \frac{w_x}{2}\right) + \left(G_z - \frac{w_z}{2}\right)\left(G_x - \frac{w_x}{2}\right)$$

$$- \left(G_z + \frac{w'_z}{2}\right)\left(G_x + \frac{w'_x}{2}\right) - \left(G_z - \frac{w'_z}{2}\right)\left(G_x - \frac{w'_x}{2}\right)\Bigg]^2$$

$$\times \sqrt{\frac{2T}{m}}\, w\, d\sigma\, d\mathbf{w}\, d\mathbf{G}$$

$$= \frac{1}{16}\sqrt{\frac{2T}{m}}\int f^{(0)} f_1^{(0)} (w_x w_z - w'_x w'_z)^2 w\, d\sigma\, d\mathbf{w}\, d\mathbf{G}$$

where $\mathbf{w}'$ is the relative velocity of the particles after collision. We have

$$\mathbf{w}' = \mathbf{w} \cos \theta + \mathbf{k} w \sin \theta$$

where θ is the scattering angle, and $\mathbf{k}$ is the unit vector perpendicular to $\mathbf{w}$, which can have any direction with the same probability in the plane perpendicular to $\mathbf{w}$. We obtain

$$\overline{(w_z w_x - w'_z w'_x)^2} = 2\overline{w_z^2 w_x^2} - 2\overline{w_x w_z w'_x w'_z}$$

$$= 2\overline{w_x^2 w_z^2} \sin^2 \theta - 2w^2 \sin^2 \theta\, \overline{w_x w_z k_x k_z}$$

where the bar above the expression denotes averaging over the angles of the relative velocity $\mathbf{w}$ and over the azimuthal angle of scattering. Thus, he have

$$\gamma_{00} = \frac{1}{8}\sqrt{\frac{2T}{m}}\int f^{(0)} f_1^{(0)} (w_x^2 w_z^2 - w^2 w_x w_z k_x k_z)\sigma_2 w\, d\mathbf{w}\, d\mathbf{G}$$

where $\sigma_2 = \int (1 - \cos^2 \theta)\, d\sigma$.

Furthermore, we have

$$f^{(0)} f_1^{(0)} = \frac{N^2}{\pi^3} \exp\left(-u^2 - u_1^2\right) = \frac{N^2}{\pi^3} \exp\left(-2G^2 - w^2/2\right)$$

Integration over dG and $w^2\,dw$ yields

$$\gamma_{00} = \frac{N^2}{8}\sqrt{\frac{2T}{m}}\frac{4}{(2\pi)^{3/2}}\int_0^\infty \exp(-w^2/2)w^7\sigma_2(w)dw\overline{(n_x^2n_z^2 - n_xn_zk_xk_z)}$$

$$= 12\sqrt{\frac{T}{\pi m}}N^2\bar{\sigma}_2\overline{(n_x^2n_z^2 - n_xn_zk_xk_z)}$$

where $\bar{\sigma}_2 = \frac{1}{48}\int_0^\infty \exp(-w^2/2)w^7\sigma_2(w)\,dw$, and $\mathbf{n}$ is the unit vector parallel to $\mathbf{w}$. Averaging over the angles yields

$$\overline{n_x^2n_z^2} = 1/15, \quad \overline{n_xk_xn_zk_z} = 1/30$$

Hence we obtain

$$\gamma_{00} = \frac{6N^2}{5\sqrt{\pi}}\sqrt{\frac{T}{m}}\bar{\sigma}_2$$

and the viscosity is

$$\eta = \frac{5\sqrt{\pi Tm}}{24\bar{\sigma}_2} \tag{1.52}$$

For instance, if the scattering of atoms is isotropic and the scattering cross section $d\sigma = \frac{\pi r_0^2}{2}d(\cos\theta)$ is independent of the relative collision velocity (the rigid sphere model), the viscosity is

$$\eta = \frac{5\sqrt{Tm}}{16\sqrt{\pi}r_0^2} \tag{1.53}$$

Equations (1.35) and (1.52) yield the following important relationship between heat conductivity and viscosity:

$$\varkappa = 15\eta/4m \tag{1.54}$$

In gas dynamics the temperature is typically expressed in terms of kelvins. To convert the energy temperature units which we employ into kelvins, we should multiply the heat conductivity by the conversion factor k_B (the Boltzmann constant). Then eq. (1.54) is transformed into

$$\varkappa = 5c_V\eta/2m, \quad c_V = \frac{3}{2}k_B \tag{1.55}$$

The relationships (1.54) and (1.55) between heat conductivity and viscosity, generally speaking, are valid only in the first Chapman-Enskog approximation but they are sufficiently accurate, since this approximation gives practically reliable results.

Problem 1.19. Determine the heat conductivity in the tau approximation.

The tau approximation is based on the replacement of the collision integral I_c with the quantity $(f - f^{(0)})/\tau$, where $f^{(0)}$ is the equilibrium distribution function, and τ is the effective time of the order of the collision interval. If the system is not at equilibrium, the kinetic equation is

$$\frac{\partial f}{\partial t} = \frac{f^{(0)} - f}{\tau}, \quad f = f^{(0)} + C \exp\left(-\frac{t}{\tau}\right)$$

that is, τ proves to be the characteristic time in which the system returns to equilibrium.

If temperature gradients are present in the gas, eq. (1.31) for deviation of the distribution function from equilibrium, $f = f^{(0)}(1 + \theta)$, is transformed into

$$f^{(0)}\left(\frac{mv^2}{2T} - \frac{5}{2}\right)\mathbf{v}\nabla \ln T = -\frac{f^{(0)}\theta}{\tau}$$

Hence we have

$$\theta = -\tau\left(\frac{mv^2}{2T} - \frac{5}{2}\right)\mathbf{v}\nabla \ln T$$

and the heat flux intensity is

$$\mathbf{q} = \int \frac{mv^2}{2}\mathbf{v} f \, d\mathbf{v} = -\int \tau \frac{mv^2}{2T}\frac{v^2}{3}\nabla T\left(\frac{mv^2}{2T} - \frac{5}{2}\right) f^{(0)} \, d\mathbf{v} = -\varkappa \nabla T$$

The heat conductivity is

$$\varkappa = \int \frac{mv^2}{2T}\frac{v^2}{3}\left(\frac{mv^2}{2T} - \frac{5}{2}\right) f^{(0)}\tau \, d\mathbf{v}$$

If τ is assumed to be independent of the velocity of particles' collision, then we obtain

$$\varkappa = 5T\tau N/2m$$

This estimate contributes to understanding the physical processes involved in heat conduction. The flow of the gas particles from a given point of space and to it is of the order of $N\bar{v}$, where $\bar{v} \sim \sqrt{T/m}$ is the characteristic thermal velocity of the gas particles. However, the particles going away from the given point and coming to it have different mean energies. The temperature difference between the given point and points at a distance of the order of the mean free path λ, from which particles come to the given point, is of the order of $\lambda \nabla T$. Thus, the heat flux tending to level off the temperatures at different points in the gas is of the order of $N\bar{v}\lambda\nabla T \sim N\bar{v}^2\tau\nabla T$ and the heat conductivity $\varkappa \sim N\bar{v}^2\tau \sim$ $\sim N\dfrac{T}{m}\tau$.

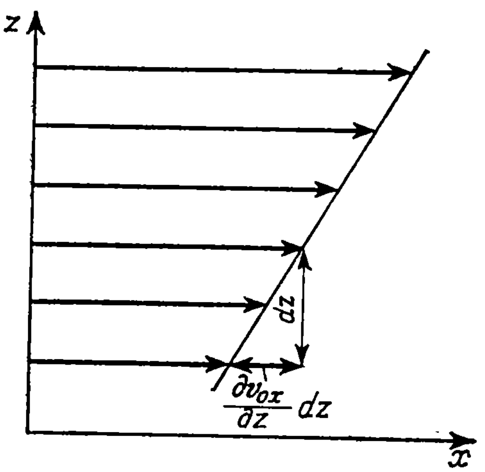

Fig. 1.2. The velocity field in the viscous gas flow.

Problem 1.20. Determine the viscosity in the tau approximation (see Problem 1.19).

Let there be a velocity gradient along the x-axis, directed along the z-axis (Fig. 1.2). Then the viscosity is found from the relation

$$F = -\eta\frac{\partial v_{0x}}{\partial z}$$

where F is the force acting on the unit area in the direction of the z-axis. The kinetic equation (1.5) in the tau approximation has the form

$$\mathbf{v}\nabla f^{(0)} = -\frac{f^{(1)}}{\tau}$$

and, since $f^{(0)} = N\left(\dfrac{m}{2\pi T}\right)^{3/2}\exp\left[-\dfrac{m(\mathbf{v}-\mathbf{v}_0)^2}{2T}\right]$, we obtain

$$v_z\frac{m(v_x - v_0)}{T}\frac{\partial v_{0x}}{\partial z}f^{(0)} = -\frac{f^{(1)}}{\tau}$$

and

$$f^{(1)} = -\frac{\partial v_{0x}}{\partial z}\frac{mv_z(v_x - v_0)}{T}\tau f^{(0)}$$

The force F is equal to the momentum transferred through the unit area per unit time. The flux of particles per unit area is Nv_z and each particle transfers the momentum $m(v_x - v_0)$. Hence we obtain

$$F = \int v_z f m(v_x - v_0)\,d\mathbf{v} = \int v_z m(v_x - v_0)f^{(1)}\,d\mathbf{v}$$

$$= -\frac{\partial v_{0x}}{\partial z}\int\frac{mv_z^2}{T}m(v_x - v_0)^2\tau f^{(0)}\,d\mathbf{v}$$

Thus, in the tau approximation the viscosity is

$$\eta = T\int \frac{mv_z^2}{T}\,\frac{m(v_x - v_0)^2}{T}\,\tau f^{(0)}\,d\mathbf{v}$$

$$= \frac{4T}{15}\int \frac{m^2(v - v_0)^4}{4T^2}\,\tau f^{(0)}\,d\mathbf{v}$$

If we assume that τ is independent of the velocity of particles' collision, we have

$$\eta = T\tau N$$

Note that in the tau approximation, if τ is independent of the collision velocity, the ratio between the heat conductivity and the viscosity is $\varkappa/\eta = 5/2m$, that is, it is smaller by a factor of 1.5 than the value obtained in the first Chapman-Enskog approximation. For the rigid sphere model ($\tau \sim 1/v$) the difference is even greater, $\varkappa/\eta = 5/4m$. This shows that the tau approximation gives only rough estimates.

The simple approach to the solution of this problem helps reveal the physical nature of this process. The flux of particles to a given point and from it is of the order of $N\sqrt{T/m}$. The particles arriving at the given point come from points at a distance of the order of the mean free path. At these points the momentum of the directed motion of particles differ from the momentum of the particles at the given point by approximately $m\frac{\partial v_{0x}}{\partial z}\lambda$. Thus, the momentum flux in the z-direction, which tends to level off the velocity of directed motion at different points, is of the order of $N\sqrt{Tm}\lambda\frac{\partial v_{0x}}{\partial z}$. Hence, we obtain the above result, that is, $\eta \sim \sqrt{Tm}N\lambda \sim T\tau N$.

Problem 1.21. Determine the heat conductivity of a gas associated with the internal degrees of freedom of the gas molecules, assuming that the rate constant for transitions between the internal degrees of freedom of the molecules during collisions is considerably smaller than the constant for elastic collision of the molecules.

In this case heat is transported by excited molecules which travel from a more heated to a less heated region where the excitation energy is dissipated. The flux of the molecules in the ith internal state is

$$\mathbf{j}_i = -\mathscr{D}N\nabla c_i$$

where $c_i = N_i/N$ is the concentration of particles in the ith state, N_i is the number density of molecules in the ith state, N is the total number density of molecules, and $\mathscr{D}$ is the diffusion coefficient of molecules, which is assumed to be independent of the state. The gradient of the concentration of molecules in a given state is caused by the temperature gradient, and we have

$$\nabla c_i = \nabla T \frac{\partial c_i}{\partial T}, \quad \mathbf{j}_i = -\mathscr{D} N \nabla T \frac{\partial c_i}{\partial T}$$

The heat flux associated with transport of energy on the internal degrees of freedom is, by definition,

$$\mathbf{q}_{\text{int}} = \sum_i \varepsilon_i \mathbf{j}_i = -\mathscr{D} N \nabla T \frac{\partial}{\partial T} (\sum_i \varepsilon_i c_i)$$

where ε_i is the excitation energy of the ith state. By definition, the mean excitation energy of the internal degrees of freedom is

$$\bar{\varepsilon} = \frac{\sum_i \varepsilon_i N_i}{\sum_i N_i} = \sum_i \varepsilon_i c_i$$

The quantity $c_{\text{int}} = \partial \bar{\varepsilon}/\partial T$ is the specific heat associated with the internal degrees of freedom. Hence we obtain

$$\mathbf{q}_{\text{int}} = -\mathscr{D} N c_{\text{int}} \nabla T = -\varkappa_{\text{int}} \nabla T$$

where $\varkappa_{\text{int}} = \mathscr{D} N c_{\text{int}}$ is the heat conductivity associated with transport of energy of the internal degrees of freedom.

If the temperature T_{int} associated with the internal degrees of freedom is not identical to the translational temperature T associated with the translational motion of the molecules, the expression for the heat conductivity should be modified to

$$\varkappa_{\text{int}} = \mathscr{D}(T) N c_{\text{int}} (T_{\text{int}}) \frac{\partial T_{\text{int}}}{\partial T} \tag{1.56}$$

Problem 1.22. Find the heat conductivity associated with transport of the energy of the vibrational and rotational degrees of freedom for a gas consisting of diatomic molecules. Use the Chapman-Enskog approximation and the rigid sphere model for elastic collisions of the molecules.

Make use of eq. (1.35) for the heat conductivity $\varkappa_{\text{el}}$ associated with the elastic collisions of molecules and eq. (1.46) for the diffusion coefficient in the Chapman-Enskog approximation. We obtain

$$\frac{\varkappa_{\text{int}}}{\varkappa_{\text{el}}} = \frac{\mathscr{D} N c_{\text{int}}}{\varkappa_{\text{el}}} = \frac{8 c_{\text{int}}}{25} = \frac{8}{25} (c_{\text{rot}} + c_{\text{vib}}) \tag{1.57}$$

Here $\varkappa_{\text{int}}$ is the heat conductivity associated with the internal degrees of freedom, and c_{int} is the specific heat associated with the internal degrees of freedom, which is in this case a sum of the rotational c_{rot} and vibrational c_{vib} specific heats.

The mean rotational energy for a diatomic molecule is T if it is assumed to be considerably larger than the difference between the rotational levels. Hence, $c_{\text{rot}} = 1$. The vibrational specific heat is, by definition,

$$c_{\text{vib}} = \frac{\partial}{\partial T} \frac{\sum_n \left(\hbar\omega n + \frac{1}{2}\right) \exp\left(-\frac{\hbar\omega n}{T}\right)}{\sum_n \exp\left(-\frac{\hbar\omega n}{T}\right)}$$

$$= \frac{\partial}{\partial T} \frac{\partial}{\partial \frac{1}{T}} \ln\left[1 - \exp\left(-\frac{\hbar\omega}{T}\right)\right]$$

$$= \left(\frac{\hbar\omega}{T}\right)^2 \exp\left(-\frac{\hbar\omega}{T}\right)\left[1 - \exp\left(-\frac{\hbar\omega}{T}\right)\right]^{-2}$$

Hence, the heat conductivity of a gas of diatomic molecules is

$$\varkappa = \varkappa_{\text{el}}\left\{1 + \frac{8}{25}\left[1 + \frac{x^2 \exp(-x)}{(1 - \exp(-x))^2}\right]\right\}, \text{ where } x = \frac{\hbar\omega}{T}$$

In this case relationship (1.57) between the viscosity and the heat conductivity is transformed into

$$\frac{\varkappa}{\eta} = \frac{15}{4m} + \frac{6}{5m}\left\{1 + \frac{x^2 \exp(-x)}{[1 - \exp(-x)]^2}\right\}$$

Problem 1.23. Find the variation of the heat conductivity caused by the dissociation of molecules into atoms in a monoatomic gas with a small admixture of diatomic molecules.

Heat transport due to dissociation occurs when a molecule migrates from a less heated to a more heated region and dissociates there. Each such molecule transports the energy of dissociation D which is much higher than the thermal energy of atoms. The flux of molecules is

$$\mathbf{j}_{\text{m}} = -\mathscr{D}_{\text{m}} N \nabla \frac{N_{\text{m}}}{N}$$

where N is the total number density of particles, N_{m} is the number density of molecules, and $\mathscr{D}_{\text{m}}$ is the diffusion coefficient for molecules in the monoatomic gas. Since each molecule transports the energy of the order of D, the heat flux is (correct to T/D)

$$\mathbf{q}_{\mathrm{dis}} = D\mathbf{j}_{\mathrm{m}} = -D\mathscr{D}_{\mathrm{m}}\nabla N_{\mathrm{m}} \qquad (N_{\mathrm{m}} \ll N)$$

The number densities of molecules N_{m} and atoms N_{a} are related by the equation $N_{\mathrm{a}}^2/N_{\mathrm{m}} = f(T)\exp(D/T)$ found in Problem 2.9 [here $f(T)$ is a power function of temperature]. Hence, with the accuracy of the order of T/D, we obtain

$$-\frac{\nabla N_{\mathrm{m}}}{N_{\mathrm{m}}} = -\frac{D}{T^2}\nabla T, \quad \mathbf{q}_{\mathrm{dis}} = -\mathscr{D}_{\mathrm{m}}N_{\mathrm{m}}\left(\frac{D}{T}\right)^2\nabla T$$

Then we obtain for the heat conductivity due to dissociation of molecules, which is determined by the equation $q_{\mathrm{dis}} = -\varkappa_{\mathrm{dis}}\nabla T$,

$$\varkappa_{\mathrm{dis}} = \mathscr{D}_{\mathrm{m}}N_{\mathrm{m}}\left(\frac{D}{T}\right)^2 \tag{1.58}$$

The heat conductivity of the monoatomic gas in this case is

$$\varkappa = \varkappa_{\mathrm{at}} + \varkappa_{\mathrm{dis}}$$

where $\varkappa_{\mathrm{at}}$ is the heat conductivity associated with heat transport by the atoms owing to collisions between them, and $\varkappa_{\mathrm{dis}}$ is the heat conductivity due to dissociation of molecules. Compare the equations for $\varkappa_{\mathrm{at}}$ and $\varkappa_{\mathrm{dis}}$:

$$\frac{\varkappa_{\mathrm{dis}}}{\varkappa_{\mathrm{at}}} \sim \frac{N_{\mathrm{m}}}{N_{\mathrm{a}}}\left(\frac{D}{T}\right)^2 \tag{1.59}$$

Since $D/T \gg 1$, estimate (1.59) shows that even if the concentration of molecules is small, their contribution to heat conductivity may be comparable to the contribution from the monoatomic gas.

Problem 1.24. Derive the transport equation for momentum in a viscous gas.

We shall make use of the Euler equation (1.18) taking into account the gas viscosity in the pressure tensor. In the coordinate system shown in Fig. 1.2 the part of the pressure tensor proportional to the gas viscosity is $P'_{xz} = -\eta\dfrac{\partial v_{0x}}{\partial z}$. In the case shown in Fig. 1.2 only the velocity component along the x-axis differs from zero and the velocity varies only in the direction z.

Since the pressure tensor is symmetric, we can write it in the following general form independent of the choice of the coordinate system:

$$P'_{ik} = -\eta\left(\frac{\partial v_{0i}}{\partial x_k} + \frac{\partial v_{0k}}{\partial x_i} + a\delta_{ik}\frac{\partial v_{0l}}{\partial x_l}\right)$$

Here summation is implied over the repeated subscripts. This expression is transformed into the one given earlier under the conditions of Fig. 1.2.

To find the factor a we shall consider the following physical picture. In this case the mechanism of viscosity is associated with friction between neighbouring gas layers travelling with different velocities. The atoms of molecules getting to the neighbouring gas regions transfer their momentum there, giving rise to the friction. If the gas is decelerated as a whole, this mechanism of viscosity is not operational. This means that the trace of the viscosity part of the pressure tensor is zero ($\sum_i P'_{ii} = 0$). Hence we obtain $a = -2/3$ and

$$P'_{ik} = -\eta\left(\frac{\partial v_{0i}}{\partial x_k} + \frac{\partial v_{0k}}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_{0l}}{\partial x_l}\right) \tag{1.60}$$

Using eq. (1.21) for the pressure tensor in terms of the gas pressure p and eq. (1.60), we obtain $P_{ik} = p\delta_{ik} + P'_{ik}$. Substitution of this expression into the Euler equation yields the transport equation for momentum,

$$\frac{\partial v_{0j}}{\partial t} + v_{0k}\frac{\partial v_{0j}}{\partial x_k} + \frac{1}{mN}\frac{\partial p}{\partial x_j} - \frac{\eta}{mN}\frac{\partial^2 v_{0j}}{\partial x_k^2} - \frac{\eta}{3mN}\frac{\partial^2 v_{0k}}{\partial x_j \partial x_k} - \frac{F_j}{m} = 0$$

where notation is the same as in eq. (1.18) and summation is implied over the repeated subscripts. The vector form of this equation is

$$\frac{\partial \mathbf{v}_0}{\partial t} + (\mathbf{v}_0\nabla)\mathbf{v}_0 + \frac{1}{mN}\nabla p - \frac{\eta}{mN}\Delta\mathbf{v}_0 - \frac{\eta}{3mN}\nabla\,\mathrm{div}\,\mathbf{v}_0 - \frac{\mathbf{F}}{m} = 0 \tag{1.61}$$

Equation (1.61) is known as the *Navier-Stokes equation.*

Problem 1.25. Derive heat transport equation for a moving gas in which heat is transported due to heat conduction mechanism.

If the volume of a monoatomic gas remains constant in the process of development of the system, the heat transport equation for the gas can be derived from a more general equation (1.20). We shall derive this equation from simple physical considerations for the conditions of the problem.

Let $\bar{\varepsilon}$ be the mean energy of a gas particle under the given conditions. Let us write the energy balance equation for a small gas volume V, taking into account that the energy is transported by the heat flux $\mathbf{q}$ through the surface of this gas region. Since the energy per unit volume is $\bar{\varepsilon}/N$ (where N is the number density of the gas particles), the energy balance equation for the volume V is

$$\frac{\partial}{\partial t}\int_V \bar{\varepsilon} N\, d\mathbf{r} = -\oint_S \mathbf{q}\, d\mathbf{s}$$

where S is the surface area corresponding to the volume V and the minus sign on the right-hand side shows that the heat flux through the surface decreases the energy accumulated in the given volume.

Making use of the Gauss theorem, we can rewrite the above equation as

$$\int\limits_V \left[\frac{\partial}{\partial t}(\bar{\varepsilon}N) + \operatorname{div}\mathbf{q}\right] d\mathbf{r} = 0$$

Since this integral is zero for any given volume, the integrand is zero. Hence we obtain the heat transport equation

$$\frac{\partial}{\partial t}(\bar{\varepsilon}N) + \operatorname{div}\mathbf{q} = 0$$

Let us write down the expression for the heat flux. Under the conditions of this problem the heat flux consists of two parts. One part is due to heat conduction and is given by eq. (1.33). The other part is associated with the particle flux $N\mathbf{v}_0$ in which each particle transfers the energy $\bar{\varepsilon}$. The resulting heat flux is $\bar{\varepsilon}N\mathbf{v}_0$.

Substituting both components of the heat flux into the heat transport equation and using the continuity equation (1.16), we obtain

$$\frac{\partial\bar{\varepsilon}}{\partial T}N\left(\frac{\partial T}{\partial t} + \mathbf{v}_0\nabla T\right) - \varkappa\Delta T = 0$$

If the moving gas is in a closed volume, then $\left.\frac{\partial\bar{\varepsilon}}{\partial T}\right|_V = c_V$, where c_V is the specific heat per one particle at constant volume (for monoatomic gases $c_V = 3/2$). Then the heat transport equation is

$$c_V N\left(\frac{\partial T}{\partial t} + \mathbf{v}_0\nabla T\right) - \varkappa\Delta T = 0 \tag{1.62}$$

Problem 1.26. (the *Rayleigh problem*). A gas is contained between two infinite parallel plates at a distance L from each other. The plates' temperatures are T_1 and T_2 ($T_2 > T_1$). The external field acts on each molecule of the gas with the force $\mathbf{F}$ which is directed from the first to the second plate, perpendicular to them. Determine the conditions at which stationary gas is at equilibrium.

If the temperature difference is large enough, heat transfer from one plate to the other occurs via motion of the gas (convection). Let us find out the conditions when such motion is possible. Assume that the gas velocity and the associated variation of temperature are small compared with the characteristic values of these quantities. Then we can find the conditions for convection regarding the convective gas motion as a perturbation of the stationary state of the gas.

We shall make use of the following steady-state equations: the continuity equation for the gas

$$\operatorname{div} N\mathbf{v}_0 = 0$$

the Navier-Stokes equation (1.61)

$$(\mathbf{v}_0\nabla)\mathbf{v}_0 + \frac{\nabla p}{MN} - \frac{\eta\Delta\mathbf{v}_0}{MN} - \frac{\eta \operatorname{grad}\operatorname{div}\mathbf{v}_0}{MN} - \frac{\mathbf{F}}{M} = 0$$

and the heat transport equation

$$c_V N\mathbf{v}_0\nabla T = \varkappa\Delta T$$

Here N is the number density of the gas particles, $\mathbf{v}_0$ is the velocity of the directed motion of the gas, p is the gas pressure, T is the temperature, η is the viscosity, $\varkappa$ is the heat conductivity, and M is the mass of a gas molecule.

Let us write each quantity in the form $A = A_0 + A'$, where A_0 is the quantity for the stationary gas and A' is the perturbation due to convection. Assuming $\mathbf{v}_0 = 0$, we obtain in the zero approximation

$$\frac{\nabla p}{MN} - \frac{\mathbf{F}}{M} = 0, \quad \Delta T = 0$$

The second equation yields the temperature distribution in the stationary gas:

$$T = T_2 - \frac{T_2 - T_1}{L}x$$

The x-axis is perpendicular to the plates; the second plate is at $x = 0$ and the first plate at $x = L$.

In the first approximation the continuity equation yields $N_0 \operatorname{div}\mathbf{v}_0 + \mathbf{v}_0\nabla N_0 = 0$. Since the density of the stationary gas is constant ($\nabla N_0 = 0$), this equation is reduced to the condition of incompressibility of the gas:

$$\operatorname{div}\mathbf{v}_0 = 0 \tag{1.63 a}$$

In the Navier-Stokes equation we shall make use of the expansion

$$\frac{\nabla p}{MN} - \frac{\mathbf{F}}{M} = \frac{\nabla p_0}{MN_0} + \frac{\nabla p'}{MN_0} - \frac{\nabla p_0}{MN_0}\frac{N'}{N_0} - \frac{\mathbf{F}}{M} = \frac{\nabla p'}{MN_0} - \frac{\mathbf{F}}{M}\frac{N'}{N_0}$$

Furthermore, we have $N' = T'\left(\frac{\partial N}{\partial T}\right)\Big|_p = \frac{T'}{T}N_0$ (T is the temperature in the stationary gas), since perturbations are caused by slow motions and if no external forces are acting on the gas in the process, the pressure is constant and the equation of state is $p = NT$. Hence, the Navier-Stokes equation can be written in the following form:

$$\nabla p' - \eta\Delta\mathbf{v}_0 - \mathbf{F}N_0\frac{T'}{T} = 0 \tag{1.63 b}$$

In the first approximation the heat transport equation (1.62) can be written as

$$-c_V N_0 v_{0x}\frac{(T_2-T_1)}{L}=\varkappa\Delta T' \tag{1.63 c}$$

Finding the solution of the system of equations (1.63) and the conditions of its existence, we can determine the conditions under which convection can occur.

Let us reduce the system of three equations (1.63) to one equation. Acting with the div operator on the second equation and using the first equation, we obtain

$$\Delta p'=\frac{FN_0}{T}\frac{\partial T'}{\partial x}$$

Here and below we assume that $(T_2-T_1)/T_2 \ll 1$, that is, variation of temperature of the stationary gas between the plates is small. Therefore, we can neglect the dependence of the gas density and temperature on the coordinate and the coordinate dependence of $\varkappa$ and η, which are functions of the temperature, in comparison with the coordinate dependence of the quantity T' which noticeably varies at a distance of the order of L.

Acting with the Laplace operator Δ on the heat transport equation and substituting the resulting expression for Δv_{0x} into the Navier-Stokes equation, we obtain

$$\frac{\partial p'}{\partial x}+\frac{\eta\varkappa L}{c_V N_0(T_2-T_1)}\Delta^2 T'-FN_0T'=0$$

Acting on this equation with the Laplace operator Δ and using the above expression for $\Delta p'$, we obtain finally

$$\frac{R}{L^4}\left(\Delta-\frac{\partial^2}{\partial x^2}\right)T'=\Delta^3 T'$$

where the dimensionless combination of the parameters of the problem

$$R=\frac{c_V F N_0^2(T_2-T_1)L^3}{\varkappa\eta T} \tag{1.64}$$

is known as the *Rayleigh number*.

It can be seen that the Rayleigh number is a characteristic parameter describing the possibility of convective motion. Only this dimensionless combination of parameters enters the equation of convective motion.

Let us formulate the boundary conditions for this problem according to which at the boundary $v_{0x}=0$, $T'=0$ and the tangential forces are zero. Since the tangential force is $\eta\frac{\partial v_{0y}}{\partial x}$ or $\eta\frac{\partial v_{0z}}{\partial x}$, the condition means that at the boundary $\partial v_{0y}/\partial x=\partial v_{0z}/\partial x=0$.

Differentiating equation $\operatorname{div} \mathbf{v}_0 = 0$ with respect to x and using the boundary conditions, we obtain $\partial^2 v_{0x}/\partial x^2 = 0$. Thus, the boundary conditions of the problem should be written in the following form: $T' = 0$, $v_{0x} = 0$, $\partial^2 v_{0x}/\partial x^2 = 0$ for $x = 0$ and $x = L$.

Let us solve the equation for T' for the above boundary conditions. Writing $T' = \exp[i(k_1 y + k_2 z)] f(x)$, we obtain

$$\frac{Rk^2 f}{L^4} = \left(k^2 - \frac{\partial^2}{\partial x^2}\right)^3 f, \quad k^2 = k_1^2 + k_2^2$$

Since $f = 0$ for $x = 0$ and $x = L$, we find $f = \text{const} \cdot \sin \varkappa x$ ($\varkappa L = \pi n$ and n is an integer). Substituting this into the equation for f, we obtain

$$R = \frac{(k^2 L^2 + \pi^2 n^2)^3}{k^2 L^2}$$

Hence we can find the minimum possible Rayleigh number corresponding to $n = 1$ and $k = \pi/L\sqrt{2}$:

$$R_{\min} = \frac{27\pi^4}{4} = 657.5$$

Under the given boundary conditions convection can occur at this and higher values of R, and for lower values of R the stationary state of the gas is stable. For other boundary conditions and other system geometry the minimum Rayleigh number at which convection occurs has different values. However, the above equations show that the Rayleigh number is the parameter whose value characterizes the convection stability of a stationary gas.

Problem 1.27. Analyze the character of gas motion in the case of free convection in the Rayleigh problem (see Problem 1.26).

For the sake of simplicity we shall assume that the velocity of a gas lies in one plane, that is, only the velocity components v_{0x} and v_{0y} differ from zero. Furthermore, assuming that convection is slow, we shall make use of the results of Problem 1.26. Considering the parameters to be independent of the coordinate z, we have $T' = \text{const} \cdot \cos ky \sin \pi x/L$. We can find v_{0x} from the heat transport equation and from the continuity equation $\partial v_{0x}/\partial x + \partial v_{0y}/\partial y = 0$ we can find v_{0y}:

$$v_{0x} = u \cos ky \cdot \sin \frac{\pi}{L} x$$

$$v_{0y} = -u \frac{\pi}{Lk} \sin ky \cdot \cos \frac{\pi}{L} x$$

Here u is a constant and k is related to the Rayleigh number:

$$R = (\pi^2 + k^2 L^2)^3 / k^2 L^2$$

Let us analyze the motion of a gas element which at a given moment is at the point (x, y). The equations of motion for this gas element are $dx/dt = v_{0x}$ and $dy/dt = v_{0y}$. Eliminating time from these equations and using the above expressions for the velocities, we obtain

$$\frac{dx}{dy} = \frac{v_{0x}}{v_{0y}} = -\frac{kL}{\pi} \frac{\tan \frac{\pi}{L} x}{\tan ky}$$

Solving this equation, we find $\ln \sin \frac{\pi}{L} x + \ln \sin ky = 0$ and hence

$$\sin \frac{\pi}{L} x \cdot \sin ky = \text{const}$$

This solution describes the trajectory of an individual gas element. Each trajectory is a closed line confined to a gas cell with the length of π/k. Motion occurs around the centre of such a cell.

Figure 1.3 shows a schematic of convection in a gas for $k = \pi/L$. In the general case these convection cells are prisms with regular polygons as bases. They are known as the *Benard convection cells.*

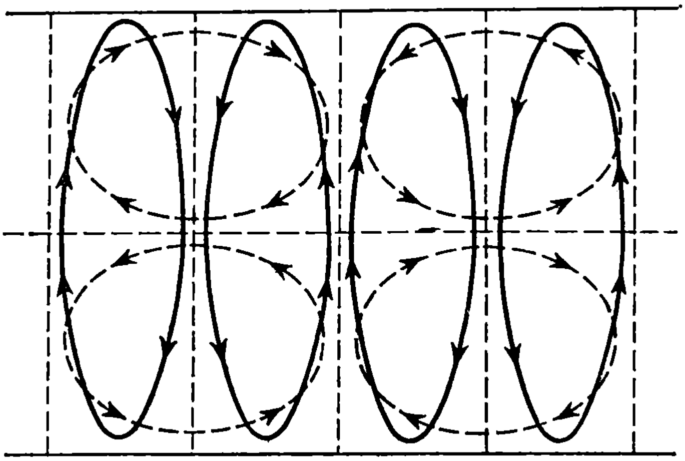

Fig. 1.3. Trajectories of the gas elements in the Rayleigh problem for $n = 1$, $k = \pi/L$.

Problem 1.28. Find the conditions for the development of convection in the Rayleigh problem for a conducting gas between the plates to which an external magnetic field is applied perpendicular to the plates.

In this problem the state of a gas must be described, apart from the continuity, Navier-Stokes and heat transport equations, by the Maxwell equations

$$\operatorname{curl} \mathbf{H} = \frac{4\pi}{c} \mathbf{j} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}, \quad \operatorname{curl} \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}, \quad \operatorname{div} \mathbf{H} = 0$$

and the Ohm's law

$$\mathbf{j} = \Sigma\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}_0\mathbf{H}]\right)$$

Here **H** and **E** are the strengths of the magnetic and electric fields, **j** is the electron current density, Σ is the conductivity of the ionized gas, and $\mathbf{v}_0$ is the velocity of the gas as a whole. We assume that the current is carried by electrons, so that $\frac{1}{c}[\mathbf{v}_0\mathbf{H}]$ is the strength of the additional electric field due to motion of electrons in the magnetic field.

Let us act on the first Maxwell equation with the operator curl. Substituting the Ohm's equation for the current density into the result and using the second and third of the Maxwell equations which have the form $\operatorname{curl}\mathbf{E} = \operatorname{div}\mathbf{H} = 0$, we obtain for the stationary case under consideration (taking into account the continuity equation $\operatorname{div}\mathbf{v}_0 = 0$)

$$\frac{c^2}{4\pi\Sigma}\Delta\mathbf{H} = (\mathbf{v}_0\nabla)\mathbf{H} - (\mathbf{H}\nabla)\mathbf{v}_0$$

Furthermore, in the Navier-Stokes equation we should take into account the force with which the magnetic field acts on the gas. This force acts directly on electrons and via them on the gas. The Lorentz force acting on the electron is $-\frac{e}{c}[\mathbf{wH}]$, where **w** is the drift velocity of the electron. The force applied to the unit volume of the gas is $-\frac{e}{c}[\mathbf{wH}]N_e = \frac{1}{c}[\mathbf{jH}]$, where N_e is the electron density, and **j** is the current density. Hence, the force of the magnetic field acting on one gas molecule is

$$\mathbf{F} = \frac{1}{Nc}[\mathbf{jH}] = \frac{1}{4\pi N}[\operatorname{curl}\mathbf{H}\mathbf{H}] = \frac{1}{4\pi N}\left\{(\mathbf{H}\nabla)\mathbf{H} - \nabla\frac{H^2}{2}\right\}$$

The Maxwell equation $\operatorname{curl}\mathbf{H} = \frac{4\pi}{c}\mathbf{j}$ has been used here. Substitution of this expression into the Navier-Stokes equation (1.61) yields

$$(\mathbf{v}_0\nabla)\mathbf{v}_0 + \frac{\nabla\left(p + \frac{H^2}{8\pi}\right)}{MN} - \frac{\eta\Delta v_0}{MN} - \frac{(\mathbf{H}\nabla)\mathbf{H}}{4\pi MN} - \frac{\mathbf{F}}{M} = 0$$

We shall determine the convection threshold as it was done above for the Rayleigh problem. We assume that in the zero approximation the gas is stationary, the temperature gradient in it is constant and the magnetic field $\mathbf{H}_0$ is spatially uniform and directed along the x-axis. Similarly to the Rayleigh problem, in the first approximation we obtain the following system of equations:

$$\operatorname{div}\mathbf{v}_0 = 0, \quad \nabla\left(p' + \frac{\mathbf{H}_0\mathbf{H}'}{4\pi}\right) - \eta\Delta\mathbf{v}_0 - \mathbf{F}N\frac{T'}{T} - \frac{H_0}{4\pi}\frac{\partial\mathbf{H}'}{\partial x} = 0$$

$$c_V N_0 v_{0x}\frac{(T_2 - T_1)}{L} = -\varkappa\Delta T', \quad \frac{c^2}{4\pi\Sigma}\Delta\mathbf{H}' + H_0\frac{\partial\mathbf{v}_0}{\partial x} = 0$$

Here $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}'$, where H' is due to convection. The first and third equations of this system are identical to the similar equations of the Rayleigh problem.

To find the convection threshold, we have to eliminate all the variables except T from these equations and to find the solution of the resulting equation using the boundary conditions. We shall use the same procedure as in the Rayleigh problem. First, act on the Navier-Stokes equation with the operator div. Since $\operatorname{div}\mathbf{v}_0 = \operatorname{div}\mathbf{H} = 0$, we obtain

$$\Delta\left(p' + \frac{H_0H'_x}{4\pi}\right) = \frac{FN_0}{T}\frac{\partial T'}{\partial x}$$

From the last two equations we obtain then

$$\Delta H'_x = -\frac{4\pi H_0\Sigma}{c^2}\frac{\partial v_{0x}}{\partial x} = \frac{4\pi H_0\Sigma\varkappa L}{c^2 c_V N_0(T_2 - T_1)}\frac{\partial T'}{\partial x}$$

Then we act on the xth component of the Navier-Stokes equation with the Laplacian Δ and substitute into the resulting equation the expression for Δv_{0x} from the heat transport equation and the above expressions for $\Delta(p' + H_0H'_x/4\pi)$ and $\Delta H'_x$ to obtain the following equation for T':

$$-\frac{FN_0}{T}\left(\Delta - \frac{\partial^2}{\partial x^2}\right)T' + \frac{\eta\varkappa L\Delta^3 T'}{c_V N_0(T_2 - T_1)} - \frac{H_0\Sigma\varkappa L}{c^2 c_V N_0(T_2 - T_1)}\Delta\frac{\partial^2 T'}{\partial x^2} = 0$$

We shall rewrite this equation in the form

$$-RL^2\left(\Delta - \frac{\partial^2}{\partial x^2}\right)T' + L^6\Delta^3 T' - M^2L^4\Delta\frac{\partial^2 T'}{\partial x^2} = 0$$

with the dimensionless parameters

$$R = \frac{c_V F N_0^2(T_2 - T_1)L^3}{\varkappa\eta T} \quad \text{(the Rayleigh number)}$$

and

$$M = \frac{H_0 L}{c}\sqrt{\frac{\Sigma}{\eta}} \quad \text{(the Hartmann number)}$$

As in the Rayleigh problem, we shall seek the solution of this equation with the boundary conditions $T' = \partial^2 T'/\partial x^2 = 0$ in the form $T' = \text{const}\cdot\exp[i(k_1 y + k_2 z)]\sin\alpha x$. The boundary condition $T' = 0$ for $x = L$ yields $\alpha = \pi n/L$, where n is an integer. Substituting the expression for T' into the equation for this quantity, we find

$$R = \frac{(k^2L^2 + \pi^2 n^2)^3}{k^2L^2} + \frac{\pi^2 n^2(\pi^2 n^2 + k^2L^2)}{k^2L^2} M^2$$

where $k^2 = k_1^2 + k_2^2$. The minimum Rayleigh number at which convection can occur corresponds to $n = 1$ and $k = k_{\text{min}}$, which is a solution of the equation

$$(k_{\text{min}}^2 L^2 + \pi^2)^2(2k_{\text{min}}^2 L^2 - \pi^2) - \pi^4 M^2 = 0$$

Hence

$$R_{\text{min}} = \frac{2(k_{\text{min}}^2 L^2 + \pi^2)^3}{\pi^2}$$

These solutions show that magnetic field stabilizes the stationary state of the gas. For small Hartmann numbers we have

$$R_{\text{min}} = \frac{27\pi^4}{4} + 3\pi^2 M^2$$

For large Hartmann numbers R_{min} tends to

$$R_{\text{min}} = \pi^2 M^2$$

Problem 1.29. Consider a discharge occurring between two infinite parallel plates in a molecular gas. The vibrational temperature of the gas is much higher than the translational temperature. Assume that the rate constant k_r of vibrational relaxation strongly depends on temperature. The distance between the plates is $2l$. Find the threshold of thermal explosion, that is, the unstable state at which heat transport via heat conduction is insufficient and the temperature in the gas increases explosively, resulting in levelling off of the vibrational and translational temperatures.

In this problem heat is liberated in the discharge gap owing to vibrational relaxation and transported to the plates via heat conduction mechanism. We have to determine the conditions under which heat evolution is so intense that no steady-state solution of the energy balance equation exists. The best approach is to find the steady-state solution and the conditions under which it does not exist.

Write down the energy balance equation for the discharge gap as

$$\varkappa \frac{d^2T}{dx^2} + f(T) = 0$$

where $\varkappa$ is the heat conductivity of the gas, x is the axis perpendicular to the plates, and the coordinate origin is at the centre of the gap. The quantity $f(T) = N^*Nk_r\hbar\omega$ is the heat evolved per unit volume per unit time (N^* is the number density of the vibrationally excited molecules, N is the number density of the atoms or molecules, the collisions with which result in vibrational relaxation, and $\hbar\omega$ is the energy of the vibrational quantum). Since the vibrational temperature is high in comparison with the translational temperature, we do not take into account the reverse process resulting in excitation of molecules. It is important that the rate constant $k_r(T)$ of vibrational relaxation strongly depends on temperature. Let us define a new quantity $\beta = d[\ln k_r(T)]/dT$ so that in a narrow temperature range we have $f(T) = f(T_0)\exp[-\beta(T_0 - T)]$, where T_0 is the temperature of the gas at the centre of the discharge gap.

Let us solve the energy balance equation. Since evolution of heat occurs only at the central region of the gap owing to its strong dependence on temperature, we can replace $f(T)$ in the equation by const$\cdot\delta(x)$. The solution of this equation with the boundary conditions $T = T_w$ for $x = l$ and $T = T_0$ for $x = 0$ is

$$T = T_0 - \Delta T x/l, \quad \text{where} \quad \Delta T = T_0 - T_w$$

We have assumed here that the dependence of the heat conductivity on temperature can be ignored in the given temperature range.

Now find ΔT taking into account the gas temperature distribution in the gap. Integrate the energy balance equation over the gap:

$$\int_0^l f(T)\,dx = \frac{\varkappa \Delta T}{l}$$

On the other hand, using the above dependence of the rate constant of vibrational relaxation on the gas temperature we obtain

$$\int_0^l f(T)\,dx = \int_{T_w}^{T_0} f(T_0)\exp[-\beta(T_0 - T)]\frac{l\,dT}{\Delta T} = \frac{lf(T_0)}{\Delta T\cdot\beta} \equiv \frac{\varkappa\Delta T}{l}$$

Hence

$$\Delta T^2 = \frac{l^2 f(T_0)}{\beta\varkappa}$$

The assumption that heat is evolved at the centre of the discharge gap implies that the following condition should be satisfied:

$$\beta\,\Delta T = l\sqrt{\frac{\beta f}{\varkappa}} > 1$$

Let us analyze the derived relationship as a function of the temperature at the centre of the gap. The left-hand side of the relationship is zero for $T_0 = T_w$ and increases as a power function with increasing T_0. The right-hand side is finite at $T_0 = T_w$ and increases exponentially with increasing T_0, that is, with a much higher rate than the left-hand side. Therefore, under certain conditions this relationship is not satisfied, since for any $T_0 - T_w$ the right-hand side is larger than the left-hand side. For instance, this is the case when the heat conductivity $\varkappa$ tends to zero. The lack of steady-state solution in this case means physically that the evolved heat remains in the gas if it cannot be removed by heat conduction mechanism. This leads to increasing gas temperature which, in its turn, results in increased evolution of heat owing to vibrational relaxation.

Let us find the threshold of heat instability. At the threshold the curves corresponding to the left-hand and right-hand sides of the above relationship have one common tangency point at which their derivatives are identical:

$$2\Delta T = l^2 \frac{df(T_0)}{dT} \cdot \frac{1}{\beta\varkappa} = l^2 f(T_0)/\varkappa$$

Hence, the threshold of heat instability is

$$l^2 \beta f(T_0)/4\varkappa = 1$$

This threshold can be conveniently expressed in terms of the wall temperature. Using the expression $f(T_0) = f(T_w) \exp(\beta\Delta T)$ and deriving from the above equations $\beta\Delta T = \beta l^2 f(T_0)/2\varkappa = 2$, we find the condition of development of heat instability in the following form:

$$l^2 \beta f(T_w) = 4e^2 \varkappa$$

(here e is the Napierian base). It can be seen that this condition primarily depends on the wall temperature.

§ 1.3. THE FOKKER-PLANCK EQUATION

The Fokker-Planck equation describes the state of a test particle travelling in a gas if one collision with the gas particles has only a weak effect on the state of the test particle. Let the state of the test particle be described by one continuous variable x and $W(t_0, x_0; t, x)$ be the probability that the particle which is in the state x_0 at the moment t_0 is in the state x at the moment t; the probability satisfies the normalization condition

$$\int W(t_0, x_0; t, x)\, dx = 1 \tag{1.65}$$

According to the definition of probability, it satisfies the *Smoluchowski integral equation*

$$W(t_0, x_0; t+\tau, x) = \int W(t_0, x_0; t, z) W(t, z; t+\tau, x)\, dz \tag{1.66}$$

If a single change in the state of a particle is associated with a small variation of x in comparison with a certain characteristic value, the Smoluchowski integral equation is reduced to the differential Fokker-Planck equation

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x}(AW) + \frac{\partial^2}{\partial x^2}(BW) \tag{1.67}$$

where

$$A(x, t) = \lim_{\tau\to 0} \frac{1}{\tau} \int (z-x) W(t, x; t+\tau, z)\, dz \tag{1.68 a}$$

$$B(x, t) = \lim_{\tau\to 0} \frac{1}{2\tau} \int (x-z)^2 W(t, x; t+\tau, z)\, dz \tag{1.68 b}$$

To reveal the physical picture, write down the Fokker-Planck equation in the form of a continuity equation

$$\frac{\partial W}{\partial t} = -\frac{\partial j}{\partial x} \tag{1.69}$$

where the particle flux

$$j = AW - B\frac{\partial W}{\partial x} \tag{1.70}$$

is a sum of two components. The first component (the hydrodynamic flux) is proportional to the density of states (or the number of the test particles) and the second component (the diffusion flux) is determined by the gradient of the density of states (the gradient of the number density of the particles).

Let us write the Fokker-Planck equation for the case when the normalization condition is more general than eq. (1.65), that is,

$$\int \rho(x) W(t_0, x_0; t, x)\, dx = 1$$

This can be done by replacing in eq. (1.67) the quantity W by ρW; the resulting equation is

$$\frac{\partial W}{\partial t} = -\frac{1}{\rho}\frac{\partial}{\partial x}(AW\rho) + \frac{1}{\rho}\frac{\partial^2}{\partial x^2}(BW\rho) \tag{1.71}$$

This leads to new definitions for A and B:

$$A(x, t) = \lim_{\tau\to 0}\frac{1}{\tau}\int (z-x)\rho(x)W(x, t;\ z, t+\tau)\,dz \tag{1.72 a}$$

$$B(x, t) = \lim_{\tau\to 0}\frac{1}{2\tau}\int (z-x)^2\rho(x)W(x, t;\ z, t+\tau)\,dz \tag{1.72 b}$$

Finally, the multidimensional Fokker-Planck equation can be written by analogy to the one-dimensional one as

$$\rho\frac{\partial W}{\partial t} = -\sum_k \frac{\partial}{\partial x_k}(A_k\rho W) + \sum_{k,l}\frac{\partial^2}{\partial x_k\,\partial x_l}(B_{kl}\rho W)$$

where

$$\int W\rho \prod_k dx_k = 1$$

and

$$A_k = \lim_{\tau\to 0}\frac{\overline{z_k - x_k}}{\tau},\quad B_{kl} = \lim_{\tau\to 0}\frac{\overline{(z_k - x_k)(z_l - x_l)}}{2\tau}$$

Here x_k are the variables determining the probability density and the bar above the symbols denotes averaging:

$$\bar{a} = \int a\rho W(t, x_1, \ldots, x_n;\ t+\tau, z_1, \ldots, z_n)\prod_l dz_l$$

The Fokker-Planck equation is a linear one and, therefore, is applicable in those cases when the test particles do not change the properties of the medium in which they are studied. Another assumption used above is that the parameter x varies almost continuously, that is, at each interaction with the medium the value of this parameter does not differ much from the value at preceding interaction event.

The Fokker-Planck equation can be used for solving a wide variety of problems in which the variable varies in single transitions by small amounts.

An example of such a problem is the motion of a particle in a coordinate space when the mean free path of the particle is small compared with the range of its migration. Another class of problems deals with energy distributions of particles if variation of the particle's energy during one collision is small. Since the Fokker-Planck equation is valid if the test particles do not affect the properties of the medium, this means that the relative number of the test particles should be small.

Problem 1.30. Derive the Fokker-Planck equation for the probability density as the function of the initial state.

Write down the Fokker-Planck equation (1.71) in the form

$$\rho(z)\frac{\partial W(x,z,t)}{\partial t} = \frac{\partial}{\partial z}[A(z,t)\rho(z)W(x,z,t)] + \frac{\partial^2}{\partial z^2}[B(z,t)\rho(z)W(x,z,t)]$$

and the Smoluchowski equation (1.66) using the simplified notation $W(x,z,t) \equiv W(t_0,x;t_0+t,z)$:

$$W(x,z,t+\tau) = \int W(x,\xi,t)\rho(\xi)W(\xi,z,\tau)\,d\xi$$

The Smoluchowski equation yields

$$\frac{\partial W(x,z,t+\tau)}{\partial t} = \int \frac{\partial W(x,\xi,t)}{\partial t}\rho(\xi)W(\xi,z,\tau)\,d\xi$$

Using the Fokker-Planck equation and integrating the resulting equation by parts, we obtain

$$\begin{aligned}\frac{\partial W(x,z,t+\tau)}{\partial t} &= \int \frac{\partial}{\partial \xi}[A(\xi)\rho(\xi)W(x,\xi,t)]W(\xi,z,\tau)\,d\xi \\ &\quad + \int \frac{\partial^2}{\partial \xi^2}[B(\xi)\rho(\xi)W(x,\xi,t)]W(\xi,z,\tau)\,d\xi \\ &= -\int A(\xi)\rho(\xi)W(x,\xi,t)\frac{\partial W(\xi,z,\tau)}{\partial \xi}\,d\xi \\ &\quad + \int B(\xi)\rho(\xi)W(x,\xi,t)\frac{\partial^2 W(\xi,z,\tau)}{\partial \xi^2}\,d\xi\end{aligned}$$

On the other hand, we have

$$\frac{\partial W(x, z, t+\tau)}{\partial t} = \frac{\partial W(x, z, t+\tau)}{\partial \tau} = \int W(x, t, \xi)\, \frac{\partial W(x, \tau, \xi)}{\partial \tau}\, \rho(\xi)\, d\xi$$

Comparing the above relationships, we obtain the following equation:

$$\frac{\partial W(\xi, z, \tau)}{\partial \tau} = -A(\xi, \tau)\, \frac{\partial W(\xi, z, \tau)}{\partial \xi} + B(\xi, \tau)\, \frac{\partial^2 W(\xi, z, \tau)}{\partial^2 \xi^2}$$

Problem 1.31. Find the integral of collisions between electrons and gas particles if the variation of the momentum and energy of electrons occurs in elastic collisions with the gas particles.

Since the variation of the electron energy during elastic collision with a gas particle is small in comparison with the electron energy, the kinetic equation for energy distribution function for the electrons in the absence of external fields has the form of the Fokker-Planck equation (1.71). The integral of collision of the electrons with the gas particles can be written as

$$I_c(f) = \frac{1}{\rho(\varepsilon)}\, \frac{\partial}{\partial \varepsilon} \left[-A\rho f + \frac{\partial}{\partial \varepsilon} (B\rho f) \right]$$

where ε is the electron energy. Since the collision integral $I_c \exp(-\varepsilon/T)$ is zero in the case of thermodynamic equilibrium between electrons and the gas $[f = \exp(-\varepsilon/T)]$, we find that the relationship between the factors A and B is $A = -B/T + B' + B\rho'/\rho$, where T is the gas temperature. Thus, the collision integral for small variations of the electron energy is

$$I_c(f) = \frac{1}{\rho(\varepsilon)}\, \frac{\partial}{\partial \varepsilon} \left[\rho(\varepsilon) B(\varepsilon) \left(\frac{\partial f}{\partial \varepsilon} + \frac{f}{T} \right) \right] \tag{1.73}$$

Here

$$B(\varepsilon) = \frac{1}{2} \left\langle \int (\varepsilon - \varepsilon')^2 N v\, d\sigma(\varepsilon - \varepsilon') \right\rangle$$

N is the number density of the gas particles, v is the velocity of their collisions with the electron, and $d\sigma$ is the differential cross section for collisions of the electron with the gas particles resulting in a given variation of the electron energy. The angle brackets denote averaging over the distribution of gas particles. If electrons are free, $\rho(\varepsilon) \sim \varepsilon^{1/2}$, and to determine the collision integral we have just to find the factor $B(\varepsilon)$.

Let us calculate the collision integral (1.73) in the case when it is determined by elastic collisions between electrons and atoms. Since an elastic collision of an electron and an atom results in the variation of the electron energy of the order of $(m/M)^{1/2}\varepsilon$ (here m and M are the masses of the electron and the atom, respectively), the approach used in this problem is applicable here. We obtain

$$B = \frac{1}{2}\int (\varepsilon - \varepsilon')^2 f_a \, dv_a \, v \, d\sigma(\varepsilon \to \varepsilon') = \frac{m^2}{8}\int (v^2 - v'^2)^2 v \, d\sigma f_a \, dv_a$$

where v and v' is the electrons' velocity before and after collision with an atom, $d\sigma(\varepsilon \to \varepsilon')$ is the differential cross section for collision between the electron and the atom in which ε is the initial and ε' the final electron energy, $f_a = N_a(M/2\pi T)^{3/2} \exp(-Mv_a^2/2T)$ is the distribution function of atoms, v_a is their velocity, N_a is their number density, and M is their mass. The variation of the velocity of atoms in their collisions with electrons is small. Hence, the conservation of the relative velocity $|\mathbf{v} - \mathbf{v}_a| = |\mathbf{v}' - \mathbf{v}_a|$ of an electron and an atom in collision yields

$$v^2 - v'^2 = 2\mathbf{v}_a(\mathbf{v} - \mathbf{v}')$$

This leads to

$$B = \frac{m^2}{2}\left\langle \frac{v_a^2}{3} \right\rangle \int (\mathbf{v} - \mathbf{v}')^2 N_a v \, d\sigma = \frac{m^2 T}{2M} N_a v \int (\mathbf{v} - \mathbf{v}')^2 \, d\sigma$$

Since $|\mathbf{v} - \mathbf{v}'| = 2v \sin(\theta/2)$, where θ is the angle of scattering of an electron by an atom, we obtain

$$B = \frac{Tm^2}{M} N_a v^3 \sigma^*$$

where $\sigma^* = \int (1 - \cos\theta) \, d\sigma$ is the diffusion cross section of scattering. Here we ignored the variation of the electron energy in comparison with the energy value. Using these expression for B, we find the collision integral (1.73)

$$I_c = \frac{m}{M} T \frac{1}{v^2} \frac{\partial}{\partial v}\left[v^3 \nu \left(\frac{f}{T} + \frac{\partial f}{mv \, \partial v} \right) \right] \tag{1.74}$$

where $\nu = N_a v \sigma^*$ is the frequency of collisions of an electron with an atom.

Problem 1.32. Determine the integral collisions of the electrons with gas molecules if the loss of the energy of the electrons is determined by excitation of the molecules to the first vibrational level and the mean electron energy is considerably higher than the excitation energy $\hbar\omega$ of the vibrational level.

Since $I_c 4\pi v^2\,dv$ is the number density of the electrons getting into the velocity range dv owing to collisions, we can write it as

$$4\pi I_c v^2\,dv = -\,4\pi v^2\,dv\,\nu_{exc}(v) f(v)$$

$$+\,4\pi\left(v^2+\frac{2\hbar\omega}{m}\right) d\sqrt{v^2+\frac{2\hbar\omega}{m}}\; f\left(\sqrt{v^2+\frac{2\hbar\omega}{m}}\right)\cdot \nu_{exc}\left(\sqrt{v^2+\frac{2\hbar\omega}{m}}\right)$$

Hence

$$I_c = \frac{\hbar\omega}{m}\,\frac{1}{v^2}\,\frac{\partial}{\partial v}\left(v\,\nu_{exc} f\right) \tag{1.75}$$

Here $\nu_{exc} = N_m v \sigma_{exc}$ is the frequency of excitation of the vibrational level of a molecule by collision with an electron, N_m is the number density of the molecules, v is the electron's velocity, σ_{exc} is the cross section of excitation of the vibrational level by collision with the electron, and $\hbar\omega$ is the excitation energy of the vibrational level.

When deriving eq. (1.75) we have assumed that the electron velocity is high compared with the velocity of the molecules, its energy is high compared with vibrational quantum $\hbar\omega$, and that the molecules are in the ground vibrational state and excited only to the first vibrational state. Under such conditions the collision integral takes into account only the transitions associated with excitation of the level. The collision integral (1.75) has the form of the Fokker-Planck equation in the velocity space, taking into account only the hydrodynamic flux and the factor $A = \hbar\omega\,\nu_{exc}(v)$.

Problem 1.33. Determine the conditions of applicability of the collision integrals (1.74) and (1.75), associated with the assumption that the electron distribution function is independent of the velocity direction.

If the characteristic time τ_m during which the direction of the momentum is changed is short, then the velocity distribution function for electrons can be regarded as independent of the momentum direction after the periods much longer than τ_m. Let us show that τ_m is much shorter than the characteristic time of energy variation in the cases analyzed in the above problems.

Indeed, $\tau_m \sim \nu_{el}^{-1}$($\nu_{el}$ is the frequency of elastic scattering of an electron by an atom or a molecule). The time in which the electron energy varies noticeably is $\tau_e \sim \dfrac{1}{\nu}\left(\dfrac{\varepsilon}{\Delta\varepsilon}\right)^2$, where ν is the frequency of collisions of an electron with atoms and $\Delta\varepsilon$ is the characteristic variation of energy during one collision. Hence, if the energy losses are determined by elastic collisions between the electron and the atoms ($\nu = \nu_{el}$, $\Delta\varepsilon \sim \sqrt{m/M}\,\varepsilon$), we have

$$\frac{\tau_m}{\tau_e} \sim \frac{m}{M} \ll 1$$

If the energy losses are associated with the excitation of vibrational level ($\nu = \nu_{exc}$, $\Delta\varepsilon = \hbar\omega$), we have

$$\frac{\tau_m}{\tau_e} \sim \frac{\nu_{exc}}{\nu_{el}} \left(\frac{\hbar\omega}{\varepsilon}\right)^2 \ll 1$$

where each of the cofactors is much smaller than unity.

§ 1.4. DIFFUSION AND TRANSLATIONAL MOTION OF PARTICLES IN A GAS

If the size of a system through which test particles pass is much larger than the mean free path of these particles, the flux density of the particles, according to eq. (1.70), can be written as

$$\mathbf{j} = \mathbf{w}N - \mathscr{D}\nabla N \tag{1.76}$$

where N is the number density of the test particles, and $\mathbf{w}$ is the velocity of their translational motion. According to eq. (1.68), the diffusion coefficient is

$$\mathscr{D} = \lim_{\tau\to 0} \frac{\overline{(\mathbf{r} - \mathbf{r}')^2}}{2\tau}$$

The hydrodynamic flux of particles is caused by external forces. If the external force is small, the drift velocity $\mathbf{w}$ of a test particle is proportional to the acting force $\mathbf{F}$:

$$\mathbf{w} = \frac{1}{e} K\mathbf{F} \tag{1.77}$$

The proportionality factor K is known as the *mobility* of particles and is independent of the force $\mathbf{F}$ if it is small.

If the test particle travels in a gas in the absence of external fields, the probability of finding it at a given point of space is determined, according to the Fokker-Planck equation (1.67), by the diffusion equation

$$\frac{\partial W}{\partial t} = \mathscr{D}\Delta W \tag{1.78}$$

Here $W(\mathbf{r}, t)$ is the probability that a test particle is at the point $\mathbf{r}$ at the moment t. Diffusion of a test particle is determined by its collisions with gas particles. The trajectory of a test particle is a broken line with collisions corresponding to the angles of the line. This type of motion is often referred to as the *Brownian motion*.

Problem 1.34. Derive the diffusion equation (1.78) for a test particle travelling in a stationary gas in the absence of external fields.

Let us use the Fokker-Planck equation (1.67) for a test particle. Since the motion of the test particle is not directed, the equation for the probability $W(\mathbf{r}, t)$ that the particle is at the moment t at the point $\mathbf{r}$ is

$$\frac{\partial W}{\partial t} = \frac{\partial^2}{\partial x^2}(B_{xx}W) + \frac{\partial^2}{\partial y^2}(B_{yy}W) + \frac{\partial^2}{\partial z^2}(B_{zz}W)$$

By definition, we have

$$B_{xx} = \lim_{\tau\to 0}\frac{\overline{(x - x')^2}}{2\tau}, \quad B_{yy} = \lim_{\tau\to 0}\frac{\overline{(y - y')^2}}{2\tau}, \quad B_{zz} = \lim_{\tau\to 0}\frac{\overline{(z - z')^2}}{2\tau}$$

where x, y, and z are the coordinates of the test particle at the starting moment, x', y' and z' are the coordinates at the moment τ, and the bar above the symbols denotes averaging over all possible displacements of the particle.

Since in our system there is no preferred direction, all the factors in the equation are identical:

$$B_{xx} = B_{yy} = B_{zz} \equiv \mathscr{D}(\mathbf{r}, t) = \lim_{\tau\to 0}\frac{\overline{(\mathbf{r} - \mathbf{r}')^2}}{6\tau}$$

Note that this relationship is valid if nonuniformities in the gas are small, so that variation of the above factors at distances of the order of the mean free path of the test particles are small. Then the Fokker-Planck equation is

$$\frac{\partial W}{\partial t} = \Delta(\mathscr{D}W)$$

If the diffusion coefficient $\mathscr{D}$ is independent of the coordinate, this equation is reduced to the diffusion equation

$$\frac{\partial W}{\partial t} = \mathscr{D}\Delta W$$

Here the probability that a particle is at a given point at a given moment is normalized to unity:

$$\int W(\mathbf{r}, t)\, d\mathbf{r} = 1$$

Problem 1.35. Find the mean square distance travelled by a Brownian particle during the time t.

The probability $W(\mathbf{r}, t)$ that a given particle is at the point $\mathbf{r}$ at the moment t satisfies the diffusion equation (1.78), $\mathscr{D}\Delta W = \partial W/\partial t$, where $\mathscr{D}$ is the diffusion coefficient. At the starting moment the particle was at the origin of the coordinates.

Since the problem is symmetric, the probability W is independent of angles and we can write the diffusion equation as

$$\mathscr{D}\frac{1}{r}\frac{\partial^2(rW)}{\partial r^2} = \frac{\partial W}{\partial t}$$

Next, we shall make use of the normalization condition according to which the total probability that the particle is somewhere in space is unity, that is, $\int\limits_0^\infty W\cdot 4\pi r^2\,dr = 1$. Integrating this twice by parts, we obtain

$$\frac{1}{6}\int\limits_0^\infty \frac{\partial^2(rW)}{\partial r^2}\,4\pi r^3\,dr = 1$$

Using the diffusion equation, we obtain from the above

$$\frac{1}{6}\int\limits_0^\infty \frac{\partial^2(rW)}{\partial r^2}\,4\pi r^3\,dr = \frac{1}{6\mathscr{D}}\int\limits_0^\infty \frac{\partial W}{\partial t}\,4\pi r^4\,dr = \frac{1}{6\mathscr{D}}\frac{d\overline{r^2}}{dt} = 1$$

Here $\overline{r^2}$ is the mean square distance. Integrating this equation with the initial condition $\overline{r^2}(0) = 0$ (the particle is at the origin of coordinates at the starting moment), we find

$$\overline{r^2} = 6\mathscr{D}t$$

Problem 1.36. Derive the relationship between the diffusion coefficient $\mathscr{D}$ of a test particle in a gas and the mobility K of the particle in the case of weak external fields (the Einstein equation).

The number density of test particles at the thermodynamic equilibrium with the gas at a given point of space is given by the Boltzmann distribution (1.10), $N = N_0 \exp(-U/T)$, where U is the potential energy of the external field, and T is the gas temperature. The external field gives rise to a nonzero gradient of the number density of the test particles $\nabla N = \dfrac{1}{T}\mathbf{F}N$, where $\mathbf{F} = -\nabla U$, resulting in the development of the diffusion particle flux. The total flux of the test particles is

$$\mathbf{j} = K\mathbf{F}N - \mathscr{D}\nabla N = \mathbf{F}N\left(\frac{K}{e} - \frac{\mathscr{D}}{T}\right)$$

is zero at equilibrium. Hence we obtain the relationship between the mobility and the diffusion coefficient known as the Einstein equation:

$$K = \frac{e\mathscr{D}}{T} \tag{1.79}$$

This relationship is valid for weak external fields for which the variation of the potential of an external field at a distance of the order of the mean free path is much smaller than the characteristic kinetic energy of the gas molecules. Under such conditions the external field does not disturb the thermodynamic equilibrium of the particles and the medium in which they travel, as was assumed in the derivation of the Einstein equation.

Problem 1.37. Find a solution of the diffusion equation (1.78) for a particle moving in an infinite medium with the initial condition $W(\mathbf{r}, 0) = W_0(\mathbf{r})$.

First, solve the diffusion equation (1.78) with the initial condition $W_0(\mathbf{r}) = \delta(\mathbf{r})$.

Owing to the symmetry of the problem, the probability W does not depend on angles and eq. (1.78) can be rewritten as

$$\frac{\partial r\,W}{\partial t} = \mathscr{D}\frac{\partial^2 r\,W}{\partial r^2} \tag{1.80}$$

Equation (1.80) can be satisfied with the so-called self-similar solution. Dimensional analysis suggests that the solution of eq. (1.80) depends on only one variable, r^2/t. Define a new variable $\xi = r^2/\mathscr{D}t$. The normalization condition $\int W \cdot 4\pi r^2\,dr = 1$ shows that W can be written as

$$W = \frac{1}{t^{3/2}} V(\xi)$$

The resulting equation for V is

$$4\xi V'' + 6V' + \xi V' + \frac{3}{2} V = 0$$

The solution of this equation $[V = C\exp(-\,\xi/4)]$, which is finite at infinity, yields

$$W = \frac{1}{(4\pi\mathscr{D}t)^{3/2}} \exp(-\,r^2/4\mathscr{D}t) \tag{1.81}$$

This solution can be extended to the case of an arbitrary initial condition. If at the moment $t = t_0$ the particle is at the point $\mathbf{r}_0$, so that $W = \delta(\mathbf{r} - \mathbf{r}_0)$, then eq. (1.81) for the probability that the particle is at the point $\mathbf{r}$ at the moment t yields

$$g(\mathbf{r}_0, t_0;\ \mathbf{r}, t) = [4\pi\mathcal{D}(t - t_0)]^{-3/2} \exp\left[-\frac{(\mathbf{r} - \mathbf{r}_0)^2}{4\mathcal{D}(t - t_0)}\right] \tag{1.82}$$

If at the starting moment $t = t_0$ the spatial distribution of particles is determined by the function $W_0(\mathbf{r})$, then the probability

$$W(\mathbf{r}, t) = \int W_0(\mathbf{r}')\, g(\mathbf{r}', t_0;\ \mathbf{r}, t)\, d\mathbf{r}'$$

This solution follows directly from the linearity of the equation. The function g given by eq. (1.82) is the Green's function for eq. (1.78). It should be noted that this treatment is valid for an infinite medium, when the boundary conditions do not affect the probability that a particle is in the medium.

The Green's function (1.82), that is, the probability that the particle travels the distance $|\mathbf{r} - \mathbf{r}_0|$ during the time $\tau = t - t_0$, depends only on the time interval and the distance between the points under consideration. This function can be represented as the product of three independent functions describing the displacement along each axis:

$$g(\mathbf{r} - \mathbf{r}_0, \tau) = g(x - x_0, \tau) g(y - y_0, \tau) g(z - z_0, \tau) \tag{1.83}$$

Here

$$g(x - x_0, \tau) = (4\pi\mathcal{D}\tau)^{-1/2} \exp\left[-(x - x_0)^2/4\mathcal{D}\tau\right]$$

and two other functions have a similar form.

The mean squared displacement of the particle during the given time is

$$\overline{\mathbf{r}^2} = \frac{\int_0^\infty r^4 \exp(-\alpha r^2)\, dr}{\int_0^\infty r^2 \exp(-\alpha r^2)\, dr}, \qquad \text{where } \alpha = \frac{1}{4\mathcal{D}t}$$

Hence we obtain

$$\overline{r^2} = -\frac{d}{d\alpha} \ln \int_0^\infty \exp(-\alpha r^2)\, r^2\, dr = -\frac{d}{d\alpha} \ln \frac{1}{\alpha^{3/2}} = \frac{3}{2\alpha} = 6\mathcal{D}t \tag{1.84}$$

Problem 1.38. Derive eq. (1.83), taking into account the fact that the magnitude and direction of each displacement of a particle between two successive collisions do not depend on the previous collisions, that is, displacements of the particle are random.

We have to find the function $g(x, t)$, that is, the probability that the particle travels the distance x during the time t. Let the number of collisions during this time be n and $\varphi(x_i)$ be the probability that between the ith and the $(i+1)$th collisions the particle was displaced by the distance x_i $\left(\int \varphi(x_i)\,dx_i = 1\right)$. Since collisions are random, the probability $\varphi(x_i)$ does not depend on the number of collisions. The probability $g(x)$ is given by

$$g(x)\,dx = \int \prod_i \varphi(x_i)\,dx_i$$

Here $x = \sum_{i=0}^{n} x_i$.

Let us define the characteristic functions

$$F(p) = \int_{-\infty}^{+\infty} g(x) \exp(ipx)\,dx, \qquad f(p) = \int_{-\infty}^{+\infty} \varphi(x_i) \exp(ipx_i)\,dx_i \tag{1.85}$$

for which we have, using the inverse transformation,

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(p) \exp(-ipx)\,dp, \qquad \varphi(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(p) \exp(-ipx_i)\,dp$$

Equation (1.85) yields

$$f(0) = \int_{-\infty}^{+\infty} \varphi(x_i)\,dx_i = 1$$

$$f'(0) = i \int_{-\infty}^{+\infty} x_i \varphi(x_i)\,dx_i = i\bar{x}_i \tag{1.86}$$

$$f''(0) = -\int_{-\infty}^{+\infty} x_i^2 \varphi(x_i)\,dx_i = -\overline{x_i^2}$$

where $\bar{x}_i$ and $\overline{x_i^2}$ are the mean displacement and the mean squared displacement during the time between two successive collisions.

The characteristic functions are convenient because $f(p)$ is independent of the collision number. We obtain

$$F(p) = \int_{-\infty}^{+\infty} \exp\,(ip\sum_i x_i)\prod_i \varphi(x_i)\,dx_i = f^n(p)$$

and hence

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f^n(p)\exp\,(-ipx)\,dp = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\,(n\ln f - ipx)\,dp \quad (1.87)$$

If the number n of collisions is large, the above integral is determined by small values of p, so that f can be expanded into a series with $p \to 0$. We obtain

$$\ln f = \ln\left[1 - i\bar{x}_i p - \frac{\overline{x_i^2}p^2}{2}\right] \approx i\bar{x}_i p - \frac{\overline{x_i^2}p^2}{2}$$

and hence

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left[ip(n\bar{x}_i - x) - \frac{n\overline{x_i^2}p^2}{2}\right]dp$$

$$= \frac{1}{\sqrt{2\pi\Delta^2}}\cdot\exp\left[-\frac{(x-\bar{x})^2}{2\Delta^2}\right] \quad (1.88)$$

where $\bar{x} = n\bar{x}_i$, and $\Delta^2 = n\overline{x_i^2}$. If we denote $\overline{x^2} = \overline{(\sum_i x_i)^2}$, we obtain $\Delta^2 = \overline{x^2} - (\bar{x})^2$.

Thus, the result is the normal distribution. This distribution is applicable if the main contribution to integral (1.87) is made by small values of p. The expansion of $f(p)$ used above is applicable for $\bar{x}_i p \ll 1$ and $\overline{x_i^2}p^2 \ll 1$. The main contribution to the integral (1.87) is made by the range $n\overline{x_i^2}p^2 \sim 1$. Thus, the normal distribution (1.88) is applicable after a large number of collisions.

Let us relate the parameters Δ^2 and $\bar{x}$ to time. The mean displacement is $\bar{x} = wt$, where w is the drift velocity of the particles. In the coordinate system where $w = 0$ we have $\overline{x^2} = 2\mathscr{D}t$ ($\mathscr{D}$ is the diffusion coefficient). Hence $\Delta^2 = 2\mathscr{D}t$ and the normal distribution (1.88) can be written in this case as

$$g(x, t) = (4\pi\mathscr{D}t)^{-1/2}\exp\left[-\frac{(x - wt)^2}{4\mathscr{D}t}\right]$$

Problem 1.39. An atomic beam consists of two species of atoms (or isotopes) and is irradiated with monochromatic light which excites only one atom species. The photon flux is perpendicular to the atomic beam; atoms are excited in a limited spatial region and the mean number of photons $\bar{n}$ absorbed by an atom is much greater than unity. Each absorbed photon adds to an atom the momentum $p_0 = \hbar\omega/c$ ($\hbar\omega$ is the photon energy and c is the velocity of light). Subsequently, photons are isotropically emitted from the excited atoms. Determine the distribution of absorbing atoms over momentums in the direction of photon motion. (No atomic collisions occur in the process.)

This process is used for separation of isotopes and gives rise to two beams of different atom species, or isotopes. Denote the distribution function of the absorbing atoms over the transverse momentum components after absorption of n photons as $f_n(p_\perp)$, where $p_\perp$ is the atom momentum component in the direction of the photon flux. We have the following equation for the distribution function, which follows from the nature of the process:

$$f_n(p_\perp) = \int f_{n-1}(p'_\perp) f_1(p_\perp - p'_\perp)\, dp_\perp$$

Let us find $f_1(p_\perp)$. After absorption of a photon the atom acquires the momentum p_0 in the direction of the photon flux and after subsequent emission of a photon the atom acquires the momentum p_0 in any direction with equal probability. The resulting momentum in the direction of the photon flux acquired by the atom after absorption and emission of one photon is $p_\perp = p_0(1 - \cos\theta)$, where θ is the angle between the directions of motion of the absorbed and emitted photons. The distribution function for atoms over the transverse momentum components after absorption and emission of one photon is normalized to unity and is given by

$$f_1(p_\perp)\, dp_\perp = d(\cos\theta)/2$$

From the above we obtain

$$f_1(p_\perp) = \begin{cases} 1/2p_0 & \text{for } 0 < p_\perp < 2p_0 \\ 0 & \text{for } p_\perp < 0,\ p_\perp > 2p_0 \end{cases}$$

Define a dimensionless variable $x = p_\perp/2p_0$. Using the expression for $f_1(p_\perp)$, rewrite the equation for the distribution function in the form

$$f_n(x) = \int\limits_{\max(x-1,\,0)}^{\min(x,\,n-1)} f_{n-1}(x')\, dx'$$

where $n > x > 0$ and the lower limit of integration is the maximum of the numbers $x - 1$ and 0. Since the number of absorbed photons is large, the distribution of atoms over the transverse momentum components has the form of the normal distribution, so that our task now is to find the mean value of the transverse momentum component and its mean square.

From the form of the distribution function $f_1(x)$ we find $\bar{x}_1 = 1/2$ and $\overline{x_1^2} = 1/3$ ($f_1(x) = 1$ for $0 < x < 1$ and $f_1(x) = 0$ for other values of x). We use here the notation $\overline{x_n^m} = \int x^m f_n(x)dx$. The above relationship for the distribution functions yields

$$\bar{x}_n \equiv \int_0^n f_n(x)dx \cdot x = \int_0^n x\,dx \int_{\max(x-1,0)}^{\min(x,n-1)} f_{n-1}(x')\,dx'$$

By changing the limits of integration, we obtain

$$\bar{x}_n = \int_0^{n-1} f_{n-1}(x')\,dx' \int_{x'}^{x'+1} x\,dx = \int_0^{n-1} \left(x' + \frac{1}{2}\right) f_{n-1}(x')\,dx' = \bar{x}_{n-1} + \frac{1}{2}$$

Since $\bar{x}_1 = 1/2$, we obtain from this $\bar{x}_n = n/2$. In a similar way for the mean square we have

$$\overline{x_n^2} = \int_0^n f_n(x) x^2\,dx = \int_0^n x^2\,dx \int_{\max(x-1,0)}^{\min(x,n-1)} f_{n-1}(x')\,dx'$$

Changing the limits of integration we obtain

$$\overline{x_n^2} = \int_0^{n-1} f_{n-1}(x')\,dx' \int_{x'}^{x'+1} x''^2\,dx'' = \overline{x_{n-1}^2} + \bar{x}_{n-1} + \frac{1}{3} = \overline{x_{n-1}^2} + \frac{n}{2} - \frac{1}{6}$$

Since $\overline{x_1^2} = 1/3$, we obtain from this recurrence relationship

$$\overline{x_n^2} = \frac{n^2}{4} + \frac{n}{12}$$

Now let us take into account the fact that the number of photons absorbed by the atom under consideration can differ from the mean number. The probability that the atom absorbs n photons, given by the normal distribution, is

$$W_n = \frac{1}{\sqrt{2\pi\bar{n}}} \exp\left[-\frac{(n-\bar{n})^2}{2\bar{n}}\right]$$

where $\bar{n} \gg 1$ is the mean number of absorbed photons. Hence

$$\bar{x} = \int_0^\infty \bar{x}_n W_n \, dn = \frac{\bar{n}}{2}$$

$$\overline{x^2} = \int_0^\infty \overline{x_n^2} W_n \, dn = \frac{\bar{n}^2}{4} + \frac{\bar{n}}{3}$$

$$\overline{x^2} - \bar{x}^2 = \frac{\bar{n}}{3}$$

Then, using the normal distribution of atoms over the transverse momentum components, we obtain the following distribution of the absorbing atoms over the transverse momentum components after passage through the absorption region:

$$f(x) = \frac{1}{\sqrt{2\pi\Delta}} \exp\left[-\frac{(x-\bar{x})^2}{2\Delta}\right]$$

Here $\Delta = \overline{x^2} - \bar{x}^2 = \bar{n}/3$ and $\bar{x} = \bar{n}/2$.

Returning to dimensional variables, we can write the distribution of the absorbing atoms over the transverse momentum components after the passage through the absorption region in the following form:

$$F(p_\perp)\, dp_\perp = \frac{dp_\perp}{\sqrt{\frac{8}{3}\pi\bar{n}p_0^2}} \exp\left[-\frac{(p_\perp - np_0)^2}{\frac{8}{3}np_0^2}\right]$$

Here $F(p_\perp)dp_\perp$ is the probability that the transverse momentum component of atoms after absorption and emission of photons is in the range from $p_\perp$ to $p_\perp + dp_\perp$, and n is the mean number of absorbed photons. We assume here that the range of variation of the transverse momentum of the atoms in the beam is small compared with the acquired mean transverse momentum $\bar{n}p_0$. The half-width of the distribution function, that is, the momentum range in which the distribution function is larger than the half of its maximum is

$$\Delta p_\perp = 2\sqrt{\frac{8}{3}\bar{n}p_0^2 \ln 2} = 2.72\sqrt{\bar{n}}p_0$$

which is much smaller than the mean acquired transverse momentum of deflected atoms since $\bar{n} \gg 1$.

Problem 1.40. Electrons are ejected into the gas between two flat infinite electrodes and migrate in the uniform electric field from one electrode to the other. The diffusion coefficient for electrons is $\mathscr{D}$, their drift velocity in the electric field is w, the electron flux is j_0, and the distance between the electrodes is $L \gg \lambda$ (λ is the electron's mean free path). Find the distribution of the electrons in the space between the electrodes ignoring the interaction between the electrons.

The density of the electron current is

$$j_0 = -\mathscr{D}\frac{dN}{dx} + wN$$

where N is the electron number density. The solution of this equation with the boundary condition $N = 0$ for $x = L$ is

$$N = \frac{j_0}{w}\left\{1 - \exp\left[-\frac{w}{\mathscr{D}}(L - x)\right]\right\}$$

If $wL/\mathscr{D} \gg 1$, we can ignore diffusion in the bulk of the interelectrode space and we have $N = j_0/w$. If $wL/\mathscr{D} \ll 1$, the directed motion of electrons does not make any contribution to current and the electron number density is independent of w:

$$N = \frac{j_0}{\mathscr{D}}(L - x)$$

Problem 1.41. Electrons generated elsewhere are fed to a gas flowing with the velocity v. The diffusion coefficient of electrons in the gas is $\mathscr{D}$ and the number of electrons generated per unit time is α. Find the spatial distribution of electrons.

According to eq. (1.83), the distribution function for electrons at the moment T after the beginning of electron generation is

$$f(\rho, x) = \int_0^T \frac{1}{(4\pi\mathscr{D}t)^{3/2}} \exp\left[-\frac{\rho^2 + (x - vt)^2}{4\mathscr{D}t}\right] dt$$

if the electron source is at the origin of coordinates and the x-axis is parallel to the gas velocity. The number of electrons at the distance ρ from the axis is

$$f(\rho) = \int_{-\infty}^{+\infty} f(\rho, x)\, dx = \int_0^T \frac{1}{4\pi\mathscr{D}t} \exp\left(-\rho^2/4\mathscr{D}t\right) dt = -\frac{1}{4\pi\mathscr{D}} Ei\left(-\frac{\rho^2}{4\mathscr{D}T}\right)$$

that is, it decreases monotonically with the increasing distance from the axis. Here $Ei(x)$ is the integral exponential function.

Problem 1.42. An ion beam is injected into a gas between two electrodes and travels in the uniform electric field. The distance between the electrodes is L, the ion drift velocity is w, the ion diffusion coefficient is $\mathcal{D}$ and $Lw \gg \mathcal{D}$. Find the ion current as a function of time and the ion distribution over the cathode surface.

According to eq. (1.83), the ion distribution function in this problem is

$$f = \frac{N}{(4\pi\mathcal{D}t)^{3/2}} \exp\left[-\frac{\rho^2 + (x - wt)^2}{4\mathcal{D}t}\right]$$

where N is the number of ions. For $x = L$ and $\tau = t - L/w \ll L/w$, we obtain

$$f = (4\pi\alpha)^{-3/2} N \exp\left(-\frac{\rho^2}{4\alpha} - \frac{w^2\tau^2}{4\alpha}\right), \quad \alpha = \mathcal{D}\frac{L}{w}$$

The ion current $I(t)$ at the cathode is

$$I(t) = w \int_0^\infty 2\pi\rho \, d\rho \, f(\rho, \tau) = wN(4\pi\alpha)^{-1/2} \exp\left(-\frac{w^2\tau^2}{4\alpha}\right)$$

$$= \frac{Nw^{3/2}}{(4\pi\mathcal{D}L)^{1/2}} \exp\left(-\frac{w^3}{4\mathcal{D}L}\tau^2\right)$$

where $\tau = t - L/w$.

Using the distribution function, we can find now the distribution of ions over the cathode surface:

$$\frac{dn}{\rho \, d\rho} = \frac{2\pi}{(4\pi\mathcal{D}t)^{3/2}} \exp(-\rho^2/4\mathcal{D}t) \int_{-\infty}^{+\infty} \exp\left[-\frac{(x-wt)^2}{4\mathcal{D}t}\right] dx = \frac{1}{2\mathcal{D}t} \exp(-\rho^2/4\mathcal{D}t)$$

The number of ions getting to the cathode at the distance ρ from the axis is

$$dn = \frac{N}{2\alpha} \exp(-\rho^2/4\alpha)\rho \, d\rho$$

Problem 1.43. The conditions are the same as in Problem 1.42 but divergence of the ion beam is determined by the electrostatic forces between ions rather than by their diffusion. Find the conditions under which this effect is predominant.

The force $\mathbf{F} = \frac{ne^2}{r^3}\mathbf{r}$ acts on an ion at the distance r from the centre so that it travels radially from the centre of the beam (here n is the number of ions which are closer to the centre than the given ion and whose distribution with respect to this centre remains spherically symmetric, and e

is the ionic charge). The ion's velocity is $\mathbf{w} = K\dfrac{ne}{r^3}\,\mathbf{r}$, where K is the ion mobility in the gas. Hence the equation of motion for the ion is

$$\frac{d\mathbf{r}}{dt} = K\frac{ne}{r^3}\,\mathbf{r}$$

The solution of this equation with the initial condition $r = 0$ for $t = 0$ is

$$\frac{r^3}{3} = Knet$$

Hence, the number of ions at the distance r from the centre of the beam at the moment t is

$$dn = \frac{r^2\,dr}{Ket}$$

If the total number of ions is N, we have

$$dn = \begin{cases} \dfrac{r^2\,dr}{Ket} & \text{for} \quad r < r_0, \\ 0 & \text{for} \quad r > r_0. \end{cases} \qquad r_0^3 = 3KNet$$

Hence, the distribution of ions along the direction of the drift velocity is

$$dn = \frac{r_0^2 - z^2}{4Ket}\,dz, \qquad |z| < r_0$$

From the above the ion current is

$$I = \frac{dn}{dt} = w\,\frac{r_0^2 - z^2}{4Ket} = \frac{3Nw}{4r_0}\left(1 - \frac{t^2}{\tau_0^2}\right), \qquad |\tau| \leqslant \tau_0$$

where $\tau_0^3 = \dfrac{2KNeL}{w^4}$, $t = \dfrac{L}{w}$, $z = wt$, $r_0 \ll L$ and the current differs from zero for $-\tau_0 < t < \tau_0$.

The ion distribution over the cathode surface is given by

$$dn = \frac{d\rho^2}{2}\,\frac{\sqrt{r_0^2 - \rho^2}}{Ket}$$

where $\rho \leqslant r_0$ and $t = L/w$. This mechanism of beam divergence is more significant than the diffusion mechanism if $r_0 \gg \sqrt{\mathscr{D}t}$, that is, if the number of ions

$$N \gg \frac{\mathscr{D}^{3/2}}{Ke}\,\frac{L^{1/2}}{w^{1/2}} \sim \frac{(\lambda L)^{1/2}}{e^2/T}\left(\frac{v_T}{w}\right)^{1/2}$$

where λ is the mean free path of the ions, T is the temperature, v_T is the thermal velocity of the ions, and w is their drift velocity under the effect of the external field.

Problem 1.44. A charged particle travels in a gas between two coaxial cylindric metal electrodes. Determine the variation of the passage time of the particle owing to its interaction with the field of the larger electrode, which is effective if the distance between the particle and the electrode is much smaller than the electrode radius.

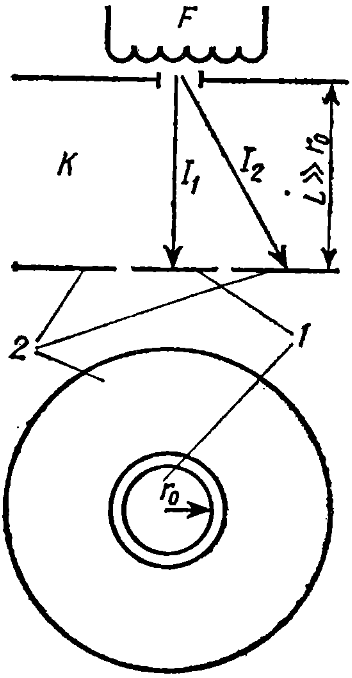

Fig. 1.4. The Townsend apparatus for measuring the electron mobilities.

At small distances from a cylindrical surface it can be assumed flat, so that the induced field acting on the particle is $e/4z^2$, where z is the distance from the surface. The electron drift velocity near the electrode surface is $w + eK/4z^2$, where w is the electron drift velocity under the effect of the external field, and K is the electron mobility. We assume that the first term becomes comparable to the second term if the distance from the cathode surface is much larger than the mean free path λ, so that the external field strength E at the electrode is small $(E \ll e/\lambda^2)$. If the distance of the particle from the electrode is smaller than the electrode radius, the equation of motion can be written as

$$\frac{dz}{dt} = w + \frac{eK}{4z^2}$$

where z is the distance from the cathode. Hence, the variation of the time of passage of the charged particle is

$$\Delta t = \int\limits_{-\infty}^{0} \left[\left(w + \frac{eK}{4z^2} \right)^{-1} - \frac{1}{w} \right] dz = -\frac{Ke}{4w^2} \int\limits_{-\infty}^{0} \frac{dz}{z^2 + \frac{eK}{4w}} = -\frac{\pi}{2w} \sqrt{\frac{Ke}{w}}$$

The following condition for the radius R of the larger electrode was used here: $R\sqrt{Kew} \gg 1$. This condition means that $\Delta t \ll L/w$, that is, the variation of the passage time is small.

Problem 1.45. The Townsend apparatus for measuring diffusion of electrons is shown in Fig. 1.4. The electrons from the emitter F pass into the drift chamber K through a slit. The electrons hit the separated anodes 1 and 2 and the anode currents I_1 and I_2 are measured. Find the ratio between these currents in the presence and absence of a magnetic field, if the size of the chamber L is much larger than the diameter $2r_0$ of anode 1.

In the absence of magnetic field the anode currents are

$$I_1 = C\int_0^{r_0} \exp\,(-\rho^2/4\mathscr{D}t)\rho\, d\rho = I_0 \exp\,(-r_0^2/4\mathscr{D}t)$$

$$I_2 = C\int_{r_0}^{\infty} \exp\,(-\rho^2/4\mathscr{D}t)\rho\, d\rho = I_0[1 - \exp\,(-r_0^2/4\mathscr{D}t)]$$

where C is a constant, $I_0 = I_1 + I_2$ is the total current, $t = L/w$, and w is the electron drift velocity. We assumed that the time of flight L/w of the electrons is considerably shorter than the time of diffusion which is of the order of $L^2/\mathscr{D}$ $(Lw/\mathscr{D} \gg 1)$, that is, the diffusion broadening of the beam is small.

In the presence of a magnetic field the electrons travel in a circle with the radius $R = wmc/eH$. If this radius is large compared with L, the horizontal displacement of the beam is $\Delta x = L^2/2R$. According to eq.(1.82), the electron distribution function with respect to the centre of the beam is

$$f = C\cdot\exp\left[-\frac{[(x-\Delta x)^2 + y^2]w}{4\mathscr{D}L}\right]$$

Hence the required ratio of the currents is

$$\frac{I_1}{I_1 + I_2} = \frac{\int_{-1}^{+1} d(\cos\vartheta)\int_0^{r_0} f\rho\, d\rho}{\int_{-1}^{+1} d(\cos\vartheta)\int_0^{\infty} f\rho\, d\rho}$$

$$= \frac{\exp\left[-\dfrac{\Delta x^2 w}{4\mathscr{D}L}\right]}{\Delta x}\int_0^{r_0} \exp\,(-\rho^2 w/4\mathscr{D}L)\sinh \frac{\rho\Delta x w}{2\mathscr{D}L} d\rho$$

Problem 1.46. Find the distribution function for the Brownian particles on a spherical surface.

This problem has practical implications, in particular, if we are interested in the direction of momentum (orbital or spin angular momentum) of a particle, whose variation after a single collision with a gas particle is small. We shall describe the momentum direction by a point on the sphere and taking $r = \text{const}$ in eq. (1.78), we can rewrite it as

$$\frac{\partial W}{\partial t} = d\left[\frac{\partial}{\partial\cos\vartheta}\left(\sin^2\vartheta\,\frac{\partial W}{\partial\cos\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2 W}{\partial\varphi^2}\right]$$

where ϑ and φ are the polar and axial angles describing the momentum direction and the diffusion coefficient d is related to the diffusion coefficient $\mathscr{D}$ in eq. (1.78) by the equation $\mathscr{D} = r^2 d$.

Expanding the distribution function $W(\vartheta, \varphi, t)$ into spherical harmonics, we find that the solution of the above equation can be written as the sum

$$W = \frac{1}{4\pi} \sum_l (2l+1) P_l(\cos\vartheta) \exp[-l(l+1)\, dt]$$

where $P_l(x)$ is the Legendre polynomial.

Let us derive two important integral relationships for the distribution function. The first relationship is for the polarization

$$P(t) = \frac{\int \cos\vartheta\, W\, do}{\int W\, do}$$

which corresponds to the mean vector projection on a given axis ($do = d(\cos\vartheta) d\varphi$ is the solid angle element). Multiply the diffusion equation by $\cos\vartheta$ and integrate the result over angles. We obtain $dP/dt = -2\, dP$ and hence, $P = P_0 \exp(-2\, dt)$, where P_0 is the polarization at the starting moment. If the starting direction of the momentum is chosen as the polarization direction ($P_0 = 1$), we obtain

$$\overline{1 - \cos\vartheta} = 1 - \exp(-2\, dt)$$

where the bar over the symbols denotes averaging over the distribution function. Hence we obtain for $t \to 0$

$$\overline{\vartheta^2} = 4\, dt$$

This equation can be used as a definition of the diffusion coefficient for a point migrating over a spherical surface.

The other relationship can be derived for the quantity $K = \overline{\sin^2\vartheta}$ by multiplying the diffusion equation by $\sin^2\vartheta$ and integrating the result over angles. We obtain $dK/dt = -6\, dK$ and

$$\overline{\sin^2\vartheta} = \frac{2}{3}[1 - \exp(-6\, dt)]$$

For $t \to 0$ this equation also yields $\overline{\vartheta^2} = 4\, dt$.

Problem 1.47. A gas contains a small admixture of molecules which intensely absorb radiation from an external source (for instance, a laser). The gas itself does not absorb the radiation. After absorption of a photon a molecule acquires the momentum in the direction of the photon motion. This effect is used for separating the admixture. The radiation beam is directed perpendicular to the flow of the gas with admixture, giving rise to a nonuniform distribution of the admixture molecules in the gas after passage through the radiation zone. Estimate the radiation energy per one admixture molecule needed to obtain a noticeable separation factor. Assume that the transverse dimension of the gas flow is larger than the photon mean free path.

Following absorption of a photon with the energy $\hbar\omega$, an admixture molecule acquires the momentum $\hbar\omega/c$ in the direction of the photon motion. Let $J_\omega\, d\omega$ be the flux of radiation with the frequency in the range from ω to $\omega + d\omega$, so that $\int J_\omega\, d\omega$ is the total radiation energy per unit area per unit time. The probability that a given molecule absorbs photons with the frequency in the range from ω to $\omega + d\omega$ per unit time is $(J_\omega d\omega/\hbar\omega)\sigma_{\text{abs}}$, where σ_{abs} is the cross section for absorption of a photon with the frequency ω. Then the force acting on an admixture molecule owing to absorption of radiation is

$$F = \frac{dp}{dt} = \frac{1}{c}\int J_\omega \sigma_{\text{abs}}\, d\omega$$

We assume here that de-excitation of the molecule occurs long before absorption of another photon.

The above force makes the admixture molecule to drift in the gas. To obtain a noticeable separation of the admixture a molecule should have time, while in the irradiation zone, to travel a distance comparable to the transverse dimensions of the flux L. This optimum time is

$$\tau \sim \frac{L}{w} \sim \frac{L}{bF} \sim \frac{TL}{\lambda v_T F} \sim \frac{TLc}{\lambda v_T \int J_\omega\, \sigma_{\text{abs}}\, d\omega}$$

where w is the drift velocity of the admixture molecule in the gas under the effect of the force F, $b \sim T/\lambda v_T$ is the mobility of the admixture molecule in the gas, T is the thermal energy of the molecules, v_T is the thermal velocity of the molecule, and λ is the mean free path of the admixture molecule in the gas.

The energy absorbed by one admixture molecule per unit time is $\int J_\omega \sigma_{\text{abs}}\, d\omega$, so that the energy required for the separation of one molecule is of the order of

$$\int J_\omega \sigma_{\text{abs}}\, d\omega\, \tau \sim T\frac{L}{\lambda}\frac{c}{v_T}$$

that is, much larger than the thermal energy of molecules.

CHAPTER 2

CHARGED PARTICLES IN A GAS

§ 2.1. PLASMA PROPERTIES DUE TO CHARGED PARTICLES

We consider a system consisting of neutral particles and a relatively small number of charged particles. However, the interaction between the neutral particles, as well as between the neutral and charged particles, is short-range interaction, in contrast to the Coulomb long-range interaction between the charged particles. For this reason some plasma properties are determined by the presence of charged particles, though their concentration is small compared with that of the neutral particles. These plasma properties are analyzed in this chapter.

We shall consider a plasma whose parameters satisfy the condition

$$e^6 N_e / T^3 \ll 1 \tag{2.1}$$

where e is the electron charge, N_e is the electron density, T is the electron temperature and plasma is assumed to be quasineutral. A plasma whose parameters satisfy condition (2.1) is known as the *ideal plasma*. Almost in all the cases of practical interest plasma is ideal. Condition (2.1) shows that in an ideal plasma the mean energy of the free electrons ($\sim T$) is much higher than the potential of interaction between them($\sim e^2 N_e^{1/3}$) at medium distances. Thus, a charged particle in the ideal plasma most of the time behaves as a free particle and travels along straight-line trajectories.

Problem 2.1. Evaluate the decrease in the ionization potential of an atom placed into a plasma.

The decrease in the ionization potential of an atom is caused by two factors. First, the presence of freee ions makes it possible for a fixed electron to jump from one ion to another and if such jump occurs freely, this type of excitation concerns the total system rather than a single atom. Second, collisions between an electron and an excited atom result in transitions between neighbouring levels. If the frequency of such transitions is higher than the difference between the frequencies of electron circulation in the orbits corresponding to these levels, we cannot distinguish between the energies of these levels.

According to the first criterion, an atom cannot be regarded as isolated if its size r_a is of the order of the mean distance between the ions $N_i^{-1/3}$.

Relating the atomic size to the ionization potential $I(r_a \sim e^2/I)$, we find the order-of-magnitude estimate of the decrease in the ionization potential:

$$\Delta I \sim e^2 N_i^{1/3} \tag{2.2}$$

The decrease in the ionization potential owing to the second mechanism is determined by the relationship

$$N_e v\sigma \sim \Delta E/\hbar$$

where v is the electron velocity, $E = \Delta I$ is the electron energy at which the electron ceases to be fixed, ΔE is the energy difference between two neighbouring levels for an electron in the Coulomb field, and σ is the cross section of transition between two neighbouring states of an excited atom under the effect of the electron impact. Let us find a classical estimate of the cross section considering the collision between an electron and an excited atom as the collision of two free electrons. The cross section of the collision resulting in transfer of energy in the range from ε to $\varepsilon + d\varepsilon$ is

$$d\sigma = \frac{2\pi e^4}{mv^2}\frac{d\varepsilon}{\varepsilon^2}$$

(see Appendix 1). Hence, the cross section of the collision of an electron and an atom resulting in the transfer of energy higher than the energy difference between the neighbouring levels ΔE is

$$\sigma = \frac{2\pi e^4}{mv^2 \Delta E}$$

Using this equation, let us find the ionization potential of an excited atom for which condition (2.3) is satisfied. We have

$$\Delta E \sim I^{3/2}(\hbar^2/me^4)^{1/2}$$

so that the state with the electron binding energy

$$I \sim e^2 N_e^{1/3}(e^2/\hbar v)^{1/2}$$

ceases to be stationary. Since the electron velocity $v \sim \sqrt{T/m}$, the decrease in the ionization potential is

$$\Delta I \sim e^2 N_e^{1/3}(me^4/\hbar^2 T)^{1/2} \tag{2.4}$$

For a quasineutral low-temperature plasma this quantity can be seen to somewhat higher than the quantity determined by (2.2) corresponding to the first mechanism. However, for the temperatures of practical interest the difference between eqs. (2.2) and (2.4) is small. Moreover, when deriving eq. (2.4), we overestimated the transition cross section. Therefore, the actual decreases in the ionization potential due to ions (2.2) and electrons (2.4) are of the same order. Note that for an ideal plasma the decrease in the ionization potential is always much smaller than the thermal energy of electrons.

Problem 2.2. Find the potential of a point charge in an ideal plasma.

The field of a charge in a plasma is shielded by the charged plasma particles. This fact should be taken into account when writing down the Poisson equation for the field **E**:

$$\operatorname{div} \mathbf{E} = 4\pi e\rho$$

The charge density ρ, according to the Boltzmann distribution, is

$$\rho = \rho_+ - \rho_- = N_0 \exp(e\varphi/T) - N_0 \exp(-e\varphi/T) = 2N_0 \sinh \frac{e\varphi}{T}$$

where N_0 is the mean number density of charged particles, ρ_+ and ρ_- are the number densities of the ions and electrons, respectively, T is the temperature of the electrons and ions, and φ is the potential of the external field. Since $\mathbf{E} = -\nabla\varphi$, the Poisson equation can be rewritten as

$$\Delta\varphi = 8\pi e N_0 \sinh \frac{e\varphi}{T} \tag{2.5}$$

Let us assume that in the ideal plasma the condition $e\varphi/T \ll 1$ is satisfied for such distances r from the point charge to the test particle for which the right-hand side of eq. (2.5) is significant. Then the Poisson equation can be written as

$$\frac{d^2}{dr^2}(r\varphi) = \frac{8\pi N_0 e^2}{T} r\varphi$$

Solving this equation, we find the Coulomb potential at small distances from the charge

$$\varphi = \frac{e}{r} \exp(-r/r_D) \tag{2.6}$$

where

$$r_D = \left(\frac{T}{8\pi N_0 e^2}\right)^{1/2} \tag{2.7}$$

is the Debye-Hückel screening radius. Thus, the field of a charge in a plasma is screened by charged particles at a distance of the order of the Debye-Hückel radius. Equations (2.1) and (2.7) show that the Debye-Hückel radius for an ideal plasma is much larger than the mean distance between the charged particles:

$$r_D N_0^{1/3} \sim \left(\frac{T^3}{N_0 e^6}\right)^{1/6} \gg 1$$

Let us analyze the validity of the assumptions used in deriving eq. (2.6). Equation (2.5) is valid if the potential φ exhibits slight variation at distances of the order of the mean distance between charged particles. Only under such conditions the mean charge density does not depend on random arrangement of particles. This condition shows that the above result is applicable if the distance from the charge $r \gg N_0^{-1/3}$. For such distances we have

$$\frac{e\varphi}{T} \lesssim \frac{e^2}{rT} \ll \frac{e^2 N_0^{1/3}}{T} \ll 1$$

so that the above approximation of the right-hand side of eq. (2.5) is correct. Since for $r \ll r_D$ the right-hand side of eq. (2.5) is not significant, this equation and its solution are applicable for all distances from the charge in the first-order approximation of the expansion in the plasma ideality parameter $e^2 N_0^{1/3}/T \ll 1$.

Problem 2.3. A uniform electric field E_0 is applied to plasma. Analyze plasma screening of the field.

Let us make use of eq. (2.5) which in this case can be written as

$$\frac{dE}{dx} = -8\pi N_0 e \sinh\frac{e\varphi}{T}$$

where $E = E_x$ is the electric field strength and the x-axis is normal to the plasma surface. Plasma is in the region $x > 0$. Multiply this equation by $E = -d\varphi/dx$ and find the solution with the boundary condition $E, \varphi \to 0$ for $x \to \infty$. We obtain

$$E = -d\varphi/dx = \pm\sqrt{32\pi N_0 T}\sinh\frac{e\varphi}{T}$$

Since the field strength and potential decrease with increasing x, the right-hand side of the above solution should be taken with the plus sign. Hence we find

$$\tanh\frac{e\varphi}{4T} = \tanh\frac{e\varphi_0}{4T}\exp(-x/r_D)$$

where $\varphi_0 = \varphi(0)$. We assume $e\varphi_0/T \ll 1$, so that

$$\varphi = \varphi_0 \exp(-x/r_D), \qquad E = -\frac{d\varphi}{dx} = \frac{\varphi_0}{r_D}\exp(-x/r_D)$$

Using the boundary condition $E_0 = E(0)$, we obtain

$$\varphi = E_0 r_D \exp(-x/r_D), \qquad E = E_0 \exp(-x/r_D)$$

This result is valid if $e\varphi_0/T \ll 1$, that is, $eE_0 r_D/T \ll 1$, or, in other words, if the energy transferred from the field to a free charged particle at a distance of the order of the screening radius is much smaller than the thermal energy of the particle.

Problem 2.4. Find the Debye-Hückel radius if the temperatures of electrons T_e and ions T_i are not identical.

In this case eq. (2.5) can be written as

$$\Delta\varphi = 4\pi e N_0 [\exp(e\varphi/T_i) - \exp(-e\varphi/T_e)]$$

For small $e\varphi/T_{e,i}$ this equation can be approximated by

$$\Delta\varphi = \varphi/r_D^2$$

Hence the Debye-Hückel radius is

$$r_D = \sqrt{\frac{T_e T_i}{4\pi N_0 e^2 (T_e + T_i)}} \tag{2.8}$$

If $T_e = T_i = T$, eq. (2.8) is transformed into eq. (2.7).

Problem 2.5. A weakly ionized plasma is generated in a small gas region and spreads from it throughout the gas. Analyze the case of ambipolar diffusion when the density of charged particles is sufficiently high, so that plasma remains quasineutral in the process of spreading. Determine the fluxes of charged particles and the conditions of ambipolar diffusion.

The fluxes of electrons $\mathbf{j}_e$ and ions $\mathbf{j}_i$ here are given by

$$\mathbf{j}_e = -\mathscr{D}_e \nabla N_e + \mathbf{E} K_e N_e$$

$$\mathbf{j}_i = -\mathscr{D}_i \nabla N_i - \mathbf{E} K_i N_i$$

where $\mathscr{D}$, K and N are the diffusion coefficient, mobility and density of a given species, and $\mathbf{E}$ is the strength of the self-consistent electric field produced by the space charge of the plasma. The diffusion coefficient $\mathscr{D} \sim v_T \lambda$, where v_T is the thermal velocity of the particles, and λ is their mean free path. Hence, $\mathscr{D} \sim m^{-1/2}$ (m is the particle's mass), that is, the diffusion coefficient of electrons is much higher than that of ions. Therefore, if the plasma density is low and the interaction between charged particles can be ignored, electrons rapidly spread throughout the bulk of the gas. However, if the plasma density is sufficiently high, the field of ions holds electrons. In this case (the case of ambipolar diffusion) the densities of charged species and their fluxes are identical. Since $\mathbf{j}_e \ll \mathscr{D}_e \nabla N_e$ correct to $\mathscr{D}_e \nabla N_e$ we obtain $-\mathscr{D}_e \nabla N_e + \mathbf{E} K_e N_e = 0$ and $\mathbf{E} = \mathscr{D}_e \nabla N_e / K_e N_e$. The flux of charged particles here is

$$\mathbf{j} = -\mathscr{D}_i \nabla N - \mathbf{E} K N = -\mathscr{D}_a \nabla N \tag{2.9}$$

where the density of charged particles is $N = N_e = N_i$ and the ambipolar diffusion coefficient is

$$\mathscr{D}_a = \mathscr{D}_i + \frac{K_i}{K_e}\mathscr{D}_e \tag{2.10}$$

The variation of the self-consistent field potential over the mean free path of the charged particles is small compared with their thermal energy. Indeed, we have $e\mathbf{E} = e\mathscr{D}_e\nabla N_e/K_e N_e \sim T_e/L$, where L is the characteristic distance of the order of the plasma size, at which a noticeable variation of plasma properties occurs. Since $L \gg \lambda$ (λ is the mean free path), we have $eE\lambda \sim T_e\lambda/L \ll T_e$, that is, the Einstein equation (1.79) is applicable here. Using it, we find the ambipolar diffusion coefficient

$$\mathscr{D}_a = \mathscr{D}_i\,(1 + T_e/T_i) \tag{2.11}$$

The electron and ion temperatures are not necessarily identical here.

Let us find the condition for applicability of the above treatment, which can be written as

$$|N_e - N_i| \ll N$$

where $N = N_e \approx N_i$. The Poisson equation

$$\operatorname{div}\mathbf{E} = 4\pi e(N_i - N_e)$$

yields $|N_i - N_e| \sim E/4\pi eL$, where L is the plasma size. On the other hand, $\mathbf{E} = \mathscr{D}_e\nabla N_e/K_e N_e \sim T_e/eL$. Hence, $|N_i - N_e| \sim T_e/4\pi e^2L^2$, and ambipolar diffusion occurs if the plasma size L is much greater than the Debye-Hückel radius $r_D = \sqrt{T_e/8\pi Ne^2}$ for the plasma:

$$L \gg r_D \tag{2.12}$$

As an illustration, in a positive discharge column the plasma size $L \sim 1$ cm and $T_e \sim 1$ eV. In such a plasma ambipolar diffusion occurs if the plasma density is higher than 3×10^5 cm^{-3}. This is a very low density, since the gas density under normal conditions is 2.7×10^{19} cm^{-3}.

Problem 2.6. Analyze penetration of slowly varying external fields into plasma (the skin effect).

Write down the system of Maxwell equations:

$$\operatorname{curl}\mathbf{H} = \frac{4\pi}{c}\mathbf{j} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$$

$$\operatorname{div}\mathbf{H} = 0$$

where **E** and **H** are the electric and magnetic fields, and **j** is the current density. This system should be supplemented by the Ohm equation

$$\mathbf{j} = \Sigma \mathbf{E}$$

where Σ is the plasma conductivity. We assume here that the magnetic field in plasma is not very strong and the conductivity can be regarded as a scalar.

Assume that the characteristic frequency ω of variation of the external fields is low compared with the plasma conductivity Σ. Using Ohm's law in the first Maxwell equation and ignoring the derivative with respect to time, we obtain curl $\mathbf{H} = 4\pi\Sigma\mathbf{E}/c$. Using this equation, eliminate **E** from the second Maxwell equation. Using the relationship curl curl $\mathbf{A} = \nabla \operatorname{div} \mathbf{A} - \Delta\mathbf{A}$, we obtain from the third Maxwell equation

$$\frac{\partial \mathbf{H}}{\partial t} = \frac{c^2}{4\pi\Sigma} \Delta \mathbf{H}$$

This equation shows that the characteristic distance over which variation of fields is noticeable is

$$l = \left(\frac{c^2}{4\pi\omega\Sigma} \right)^{1/2}$$

This parameter is known as the skin depth.

Let us consider, for instance, a plasma with a plane boundary. The magnetic field outside the plasma is $\mathbf{H} = \mathbf{H}_0 \exp(i\omega t)$. Then the magnetic field inside plasma is described by the following equation:

$$l^2 \frac{\partial^2 \mathbf{H}}{\partial z^2} = \mathbf{H} \exp(i\pi)$$

Here the z-axis is perpendicular to the boundary of the plasma. Taking into account the boundary condition and the obvious condition that the field should not increase in the plasma, we obtain the following solution of the above equation:

$$\mathbf{H} = \mathbf{H}_0 \exp\left[i \left(\omega t - \frac{z}{l\sqrt{2}} \right) - \frac{z}{l\sqrt{2}} \right]$$

It can be seen that varying external fields of a low frequency penetrate plasma to a depth of the order of the skin depth. This phenomenon is known as the *skin effect*.

§ 2.2. STATISTICAL PHYSICS OF A WEAKLY IONIZED GAS

Problem 2.7. Find the relationship between the densities of electrons, ions and atoms in an ideal plasma under thermodynamic equilibrium (the Saha equation).

The distribution of atoms over excited states is described by the Boltzmann distribution (1.10). The continuous-spectrum states are characterized by the statistical weight $g_c = g_e g_i \int d\mathbf{r} d\mathbf{p}/(2\pi\hbar)^3$, where g_i is the statistical weight of the ion, $g_e = 2$ is the statistical weight of the electron, and $\mathbf{r}$ and $\mathbf{p}$ are the coordinate and momentum of the electron. An order-of-magnitude estimate of $\int d\mathbf{r}$ is $1/N_e$, that is, the volume per one electron (N_e is the density of free electrons) and $\int d\mathbf{p} \sim p_e^3 \sim (mT)^{3/2}$ (m is the electron mass, and p_e is the characteristic momentum of the free electron). Thus, the statistical weight of the continuous-spectrum states is

$$g_c \sim \frac{g_i}{N_e}\left(\frac{mT}{\hbar^2}\right)^{3/2}$$

and it can be very high for the ideal plasma even at low temperatures. Owing to this fact the degree of ionization of a gas becomes noticeable even at the temperatures at which the mean kinetic energy of electrons is small in comparison with the ionization potential of an atom. At these temperatures the probability to find an atom in the excited state is small.

Let us find the number of free electrons in the gas at thermodynamic equilibrium, using the Boltzmann distribution (1.10). It has been shown that the probability of excitation of atoms is small, so that atoms can be only in the ground or ionized states. Let a volume Ω contain $(i + a)$ atomic cores of a given species and $(e + a)$ electrons. These electrons can be either free or bound to the atomic cores; according to the above considerations, only one bound state of an electron and atomic core should be taken into account. Collisions between particles lead to transitions between free and bound states of electrons and atomic cores and there is a nonzero probability $W(a, e, i)$ that the system at a given moment contains a atoms, e electrons and i ions. Let $\bar{a}$, $\bar{e}$ and $\bar{i}$ be the equilibrium numbers of atoms, electrons and ions, respectively ($a, e, i \gg 1$). Then the probability W that the system contains a given number of particles of a given species has a maximum at their equilibrium numbers if regarded as a function of the number a of atoms; since $\bar{a}, \bar{e}, \bar{i} \gg 1$, this condition can be conveniently written as

$$W(\bar{a}, \bar{i}, \bar{e}) = W(\bar{a} - 1, \bar{i} + 1, \bar{e} + 1) \tag{2.13}$$

We assume that the probability of finding a given electron in a free or bound state is determined by the Boltzmann distribution. Then the probability that the system contains a atoms, i ions and e electrons is

$$W(a, i, e) = Ag_0^a C_a^{a-i} \frac{\left(\prod_{j=0}^{e} \frac{\Omega d\mathbf{p}_j}{(2\pi\hbar)^3} \exp\left[-\frac{p_j^2}{2mT} - \frac{I}{T}\right]\right)(g_e)^e (g_i)^i}{e!}$$

$$\equiv A \frac{(a+i)!}{a!i!e!} (g_0)^a (g_e)^e (g_i)^i \left[\Omega \left(\frac{mT}{2\pi\hbar^2}\right)^{3/2} \exp(-I/T)\right]^e$$

Here A is the normalization factor which is independent of a for given total numbers of atomic cores $a + i$ and electrons $a + e$, g_0 is the statistical weight of an atom in the ground state, $g_e = 2$ is the statistical weight of an electron, g_i is the statistical weight of an atom, j is the number of a free electron, I is the ionization potential of an atom, T is the electron temperature, and C_a^{a+i} is the number of ways of selecting atomic cores with bound electrons (the atomic cores are assumed to be stationary).

Since transposition of electrons does not result in a new physical state, we should select the phase space for the electron momenta in such a way that the momentum of one of the labeled electrons be always higher than the momentum of another electron. This makes it possible to ignore the identical states obtained by transposition of electrons. Therefore, in the above equation the integral is taken over the entire momentum space for each electron and the statistical weight of electrons is divided by the number of transpositions of electrons.

Denote by $N_a = \bar{a}/\Omega$, $N_i = \bar{i}/\Omega$, and $N_e = \bar{e}/\Omega$ the equilibrium densities of the ground-state atoms, electrons and ions, respectively. From eq. (2.13) we obtain for the thermodynamic equilibrium ($\bar{a}$, $\bar{e}$, $\bar{i} \gg 1$)

$$\frac{N_e N_i}{N_a} = \frac{g_e g_i}{g_a} \left(\frac{mT}{2\pi\hbar^2}\right)^{3/2} \exp(-I/T) \tag{2.14}$$

This equation is known as the Saha equation. We can readily find from it the statistical weight of the continuous-spectrum electron states which is

$$g_c = \frac{g_e g_i}{N_i} \left(\frac{mT}{2\pi\hbar^2}\right)^{3/2} \tag{2.15}$$

Problem 2.8. Derive the relationship between the densities of atomic ions, molecular ions and atoms in a weakly ionized plasma if the gas temperature T is low compared with the energy D of dissociation of the molecular ion but high compared with the energy $\hbar\omega$ of excitation of the rotational level. The equilibrium distance between nuclei in the molecular ion is r_0.

The equilibrium between the atomic and molecular ions is established in the chemical reaction

$$A^+ + B \rightleftarrows AB^+$$

and is equivalent to the ionization equilibrium analyzed in Problem 2.7:

$$A^+ + e \rightleftarrows A$$

Therefore, the relationship between the densities of the atomic ions N_{1i}, atoms N_a and molecular ions N_{2i} is similar to the Saha distribution (2.14) and has the form

$$\frac{N_{1i}N_a}{N_{2i}} = \frac{g_{1i}g_a}{g_{2i}} \left(\frac{\mu T}{2\pi\hbar^2}\right)^{3/2} \exp(-D/T)$$

where μ is the reduced mass of the nuclei, g_{1i} and g_a are the statistical weights for the atomic ion and atom, which describe the number of electronic states of these particles, and g_{2i} is the number of states of the molecular ion.

This distribution differs from the Saha distribution in that at the ionization equilibrium practically all the atoms are in the ground state, since the thermal energy of electrons is low compared with the excitation energy of the atom. In this case the opposite is true, so that the molecular ion has similar probabilities to be in many rotational and, possibly, many vibrational states. Therefore, g_{2i} is the mean number of possible states of the molecular ion.

The energy of the molecular ion which is in the vibrational level v and has the momentum J is

$$\varepsilon = \hbar\omega v + BJ(J+1)$$

where $B = \hbar^2/2\mathcal{I}$ is the rotational constant ($\mathcal{I} = \mu r_0^2$ is the moment of inertia of the moleculąı ion). Hence, the mean number of the possible states of the molecular ion is

$$g_{2i} = g_{\text{mol}} \sum_v \int (2J+1)\, dJ \exp\left[-\frac{\hbar\omega v - BJ(J+1)}{T}\right]$$

where g_{mol} is the statistical weight of the molecular ion corresponding to its electronic state, and $(2J+1)$ is the statistical weight of a given rotational state of the molecular ion. Summation over the vibrational quantum number v and integration over the rotational quantum number J yield

$$g_{2i} = g_{\text{mol}} \frac{T}{B} \frac{1}{1-\exp(-\hbar\omega/T)}$$

Hence, we obtain the following relationship for the densities of ions and atoms:

$$\frac{N_{1i}N_a}{N_{2i}} = \frac{g_{1i}g_a}{g_{\text{mol}}} \exp(-D/T) \frac{1}{2\pi r_0^2} \sqrt{\frac{\mu T}{2\pi\hbar^2}} [1 - \exp(-\hbar\omega/T)]$$

We assumed here that the nuclei of the molecular ion are not different isotopes. Otherwise, the statistical weight of the molecular ion should be halved as follows from the symmetry of the nuclear wave function for identical nuclei. Indeed, the total wave function must change its sign if the nuclei are transposed and this symmetry condition for the wave function leads to the condition that the rotational wave function have a certain type of symmetry, that is, the moment of rotation must be either only even or only odd. Therefore, half of the states cannot be realized. This leads to the following relationship between the densities of the atomic and molecular ions and atoms:

$$\frac{N_{1i}N_a}{N_{2i}} = \frac{g_{1i}g_a}{g_{\text{mol}}} \exp(-D/T) \frac{1}{2\pi r_0^2} \sqrt{\frac{\mu T}{2\pi\hbar^2}} [1 - \exp(-\hbar\omega/T)]$$

Problem 2.9. Find the relationship between the densities of atoms and diatomic molecules of these atoms. The dissociation energy of the molecule is D, the spacing of the vibrational levels is $\hbar\omega$, the equilibrium distance between the nuclei is r_0, the mass of the nuclei is M, and the gas temperature is T.

The thermodynamic equilibrium here can be described by the reaction

$$\mathrm{A} + \mathrm{A} \rightleftarrows \mathrm{A}_2$$

which is equivalent to the case analyzed in Problem 2.8. If the nuclei are identical, the statistical weight of the molecule is half that for the nuclei of different isotopes. This follows from the symmetry of the molecule if we consider the reflection of nuclei in the plane of symmetry. Using the results of Problem 2.8, we obtain the following relationship between the densities of atoms N_a and molecules N_m:

$$\frac{N_a^2}{N_m} = \frac{g_a^2}{(2)\,g_{\text{mol}}} \frac{1 - \exp(-\hbar\omega/T)}{4\pi r_0^2} \sqrt{\frac{MT}{\pi\hbar^2}} \exp(-D/T)$$

where g_a and g_{mol} are the statistical weights of the atom and molecule respectively, which describe the number of their electronic states, and (2) is 2 for the nuclei of different isotopes and 1 for identical nuclei.

Problem 2.10. Find the relationship between the densities of electrons, ions and atoms if atoms are ionized via excitation of a resonance level whose de-excitation time is τ. The resonance radiation freely leaves the system. The quenching constant for the resonance excited state is k_q; radiation from other excited states does not affect the equilibrium conditions.

Under the conditions of this problem the density N^* of the atoms in the resonance excited state is related to the electron density N_e and the ion density N_i by the Saha equation

$$\frac{N_e N_i}{N^*} = \frac{g_e g_i}{g^*} \left(\frac{mT}{2\pi\hbar^2} \right)^{3/2} \exp(-I^*/T)$$

where g^*, g_e and g_i are the statistical weights of the respective particles, T is the system's temperature, and I^* is the ionization potential of an excited atom.

Let us find the relationship between the densities of atoms in the ground state N_a and resonance excited state N^*. The following balance equation can be written for the density of excited atoms:

$$\frac{dN^*}{dt} = N_e k_{exc} N_a - N_e k_q N^* - \frac{N^*}{\tau} = 0$$

Here k_{exc} is the constant of the excitation of the atom to the resonance state. Hence we find

$$\frac{N^*}{N_a} = \frac{k_{exc}/k_q}{1 + 1/N_e k_q \tau}$$

If the excited atoms do not emit radiation ($\tau \to \infty$) the densities of atoms in the ground and excited states are related by the Boltzmann distribution

$$N^* = \frac{g^*}{g_a} N_a \exp(-\Delta E/T)$$

where ΔE is the excitation energy. Now we can find the relationship between the constants of excitation and quenching. This leads to the following relationship between the densities of atoms in the case of de-excitation by radiation:

$$N^* = \frac{g^*}{g_a} N_a \exp(-\Delta E/T) N_e k_q \tau$$

Substituting this equation into the Saha equation we obtain the following relationship between the densities of electrons and atoms in a quasineutral plasma:

$$\frac{N_e}{N_a} = \frac{g_e g_i}{g_a} k_q \tau \left(\frac{mT}{2\pi\hbar^2} \right)^{3/2} \exp(-I/T)$$

where $I = I^* + \Delta E$ is the ionization potential for an atom in the ground state.

Problem 2.11. The translational and rotational temperatures T of a diatomic molecular gas are identical. The rate of the excitation of vibrational states is high (for instance, excitation by electron impact), so that the energy accumulated at the vibrational levels is much higher than the mean thermal, or rotational, energy of the molecules. Population of the vibrational levels is determined by exchange of vibrational excitations which is of a resonance nature, that is, the sums of vibrational numbers of the molecules in the initial and final states are identical. Find the equilibrium distribution of molecules over the vibrational-rotational levels.

The process leading to the equilibrium between the vibrational levels can be described by the following equation:

$$M(v_1) + M(v_2) \rightleftarrows M(v_1') + M(v_2')$$

Here M denotes the molecule and the quantity in brackets is the vibrational quantum number.

The resonance processes for which $v_1 + v_2 = v_1' + v_2'$ are the most effective ones. The transition rate constant for the nonresonance processes is small since they involve relatively large energy exchanges between the vibrational and translational degrees of freedom. Therefore, the equilibrium between the vibrational degrees of freedom is established only via resonance processes.

Let us find the relationship between the constants of the direct $k(v_1,\ v_2 \rightarrow v_1',\ v_2')$ and reverse $k(v_1',\ v_2' \rightarrow v_1,\ v_2)$ resonance processes. The equilibrium condition yields

$$N_{v_1} N_{v_2} k(v_1,\ v_2 \rightarrow v_1',\ v_2') = N_{v_1'} N_{v_2'} k(v_1',\ v_2' \rightarrow v_1,\ v_2)$$

where N_v is the density of molecules for the respective vibrational state. If the system is at thermodynamic equilibrium with respect to all degrees of freedom then the density of molecules on each vibrational level is given by the Boltzmann distribution and

$$k(v_1,\ v_2 \rightarrow v_1',\ v_2') = k\,(v_1',\ v_2' \rightarrow v_1,\ v_2) \exp\,(\Delta E/T)$$

where ΔE is the energy difference between vibrational levels for a given transition. The rate constant of a given transition, being a parameter describing a collision event, depends on the translational temperature T rather than on populations of vibrational levels; therefore, the above relationship can also be used if the vibrational temperature is not identical to the translational temperature. Then equilibrium condition yields

$$N_{v_1} N_{v_2} = N_{v_1'} N_{v_2'} \exp\,(-\Delta E/T) \tag{2.16}$$

Write down the excitation energy of a vibrational state as

$$E_v = \hbar\omega\left(v + \frac{1}{2}\right) - \hbar\omega x_e\left(v + \frac{1}{2}\right)^2$$

where ω is the harmonic oscillator frequency, and x_e is the anharmonicity parameter. The energy difference ΔE of the process under consideration is an additive function of the anharmonic corrections ΔE_v to the vibrational energy for each of the levels involved in the transition:

$$\Delta E = \Delta E_{v_1} + \Delta E_{v_2} - \Delta E_{v_1'} - \Delta E_{v_2'}$$

Here $\Delta E_v = -\hbar\omega x_e\left(v + \frac{1}{2}\right)^2$. Thus, we can find the density of molecules N_v in a given vibrational state as a function of the translational temperature T:

$$N_v \sim \exp\left(-\Delta E_v/T\right) = \exp\left[\hbar\omega x_e\left(v + \frac{1}{2}\right)^2/T\right]$$

Denote the vibrational temperature by T_v. It should be borne in mind that in the above processes establishing equilibrium with respect to the vibrational states the total number of vibrational excitations in molecules is conserved. Therefore, we can make use of the vibrational temperature which is associated with the mean number $\bar{v}$ of vibrational excitations per one molecule given by the Planck formula

$$\bar{v} = [\exp(\hbar\omega/T_v) - 1]^{-1}$$

The density of molecules on a given vibrational level as a function of the vibrational temperature T_v is determined by the Boltzmann distribution:

$$N_v \sim \exp\left(-\hbar\omega v/T_v\right)$$

Since the population of a vibrational level depends on the translational and vibrational temperatures, we find

$$N_v = N_0 \exp\left[-\hbar\omega v/T_v + \hbar\omega x_e v(v+1)/T\right] \tag{2.17}$$

where N_0 is the density of molecules in the ground vibrational state. Hence, the density of molecules in the vibrational state v with the rotational quantum number J is

$$N_{vj} = N_0 \frac{B}{T}(2J+1) \exp\left[-\hbar\omega v/T_v + \hbar\omega x_e v(v+1)/T - BJ(J+1)/T\right] \tag{2.18}$$

where B is the rotational constant of the molecule; the translational and rotational temperatures of the molecular gas are assumed to be identical.

The population of vibrational levels as a function of the vibrational quantum number, according to eq. (2.17), is shown in Fig. 2.1. As is demonstrated by eq. (2.17), the minimum of the distribution function corresponds to the vibrational quantum number

$$v_{\min} = \frac{\hbar\omega}{2\hbar\omega x_e}\frac{T}{T_v}$$

The table below gives the values of the first factor for some molecules. It can be seen that if the vibrational temperature of molecules is a few times higher than the translational temperature ($v_{\min}$ is about 10), that is, the minimum of the distribution function can be realized.

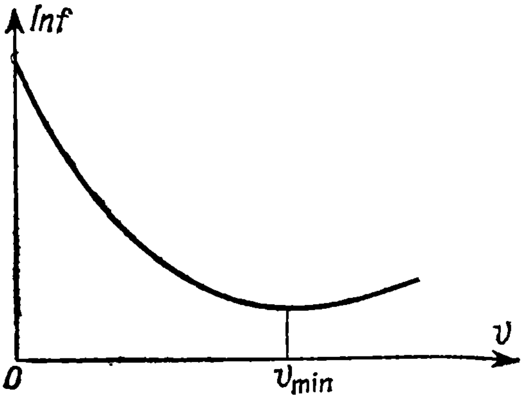

Fig. 2.1. The Treanor distribution.

Molecule	CO	H_2	N_2	NO	O_2	OH
$\hbar\omega/2\hbar\omega x_e$	82	18	84	68	66	22

Note that the form of the distribution (2.17), which is called sometimes the Treanor distribution, is determined by the fact that the value of the vibrational quantum decreases with increasing vibrational excitation. Therefore, during the exchange of vibrational quanta between colliding molecules, the formation of a higher excited molecular state proves to be more favourable, leading to increase in the logarithmic derivative of the distribution function with respect to the vibrational level number as vibrational excitation increases. This fact results even in the increase of the distribution function with increasing vibrational excitation for $v > v_{\min}$. However, as the vibrational quantum decreases, the probability of vibrational relaxation, which we ignored in derivation of distribution (2.17), grows. Ultimately, vibrational relaxation results in a sharp drop at the tail of the distribution function, that is, for high vibrational quantum numbers.

Problem 2.12. A small admixture of isotope is added to a molecular gas under the conditions of Problem 2.11. Find the distribution of the isotope molecules over the vibrational states assuming that the values of vibrational quanta for both isotopes are close.

Under the conditions of this problem, when the concentration of the added isotope is small, the admixture does not affect the distribution of the base molecules over the vibrational states, which is described by eqs. (2.17) and (2.18). The distribution of the added isotope molecules over the vibrational states is established via their collisions with the base molecules and during the exchange of vibrational quanta between colliding molecules, the total number of vibrational quanta is conserved. Hence, similarly to the solution of Problem 2.11, we replace eq. (2.16) by the following relationship between the number density N_v of the base molecules and the number density n_v of the added isotope molecules:

$$N_{v_1} n_{v_2} = N_{v_1'} n_{v_2'} \exp(-\Delta E/T)$$

Here T is the translational temperature of the gas, and ΔE is the variation of energy in the transition under consideration $(v_1, v_2 \to v_1', v_2')$.

This relationship can be rewritten as

$$\frac{n_v}{n_0} = \frac{N_v}{N_0} \exp(\Delta\hbar\omega \cdot v/T)$$

where $\Delta\hbar\omega$ is the difference between the vibrational quanta for the isotopes. Assume that the vibrational quantum for the added isotope is smaller. Using eq. (2.17), we obtain the following distribution of the added isotope molecules over the vibrational states:

$$n_v = n_0 \exp\left[-\left(\frac{\hbar\omega}{T_v} - \frac{\Delta\hbar\omega}{T}\right) v + \frac{\hbar\omega x_e v(v+1)}{T}\right]$$

It can be seen that if the difference between the vibrational temperature and the translational temperature is large, the distribution of the added isotope molecules can, in principle, correspond to the increase in the distribution function with vibrational quantum number. Vibrational relaxation changes the picture.

Problem 2.13. Calculate the mean energy of interaction between a charged test particle and the particles of an ideal plasma. Find the mean energy of a charged particle in the ideal plasma.

The interaction between two charged particles in a plasma is described by the screened Coulomb potential

$$U(r) = \frac{e^2}{r} \exp(-r/r_D)$$

where r is the distance between the particles, and r_D is the Debye-Hückel radius. If we ignore the effect of the test particle on the spatial distribution of the charged plasma particles, the mean potentials of interaction between the test particle and the charged plasma particles of both signs are identical owing to plasma quasineutrality, that is, the mean energy of interaction between plasma and a charged test particle in it is zero. We find a nonzero effect if we take into account the influence of the test particle field on the distribution of plasma particles. Then the particles of the same sign as the test particle will be at a larger distance from it than the particles of the opposite sign and the result is attraction, that is, the energy of interaction between a charged test particle and plasma is negative.

Let us calculate this energy of interaction. We have

$$\overline{U} = \int_0^\infty \frac{e^2}{r} \exp(-r/r_\mathrm{D}) \left[N_0 \exp(-e\varphi/T) - N_0 \exp(e\varphi/T)\right] d\mathbf{r}$$

where $\varphi = \frac{e}{r} \exp(-r/r_\mathrm{D})$ is the potential produced by the test particle in a plasma, T is the plasma temperature, and N_0 is the mean number density of charged particles in the plasma. According to the Boltzmann distribution, $N_0 \exp(-e\varphi/T)$ is the number density of the charged plasma particles with the same sign as the test particle, and $N_0 \exp(e\varphi/T)$ is the number density of the particles with the opposite sign. The quantity $N_0 \exp(-e\varphi/T)\, d\mathbf{r}$ is the mean number of plasma particles with the same charge sign as the test particle in the space element $d\mathbf{r}$.

Let us evaluate the above integral. The main contribution to the integral is made by the distances from the test particle of the order of the Debye-Hückel radius. Expanding the exponent at the Debye-Hückel radius, we obtain

$$\overline{U} = \int_0^\infty \frac{e^4}{r^2} \exp(-2r/r_\mathrm{D}) \frac{N_0}{T} 4\pi r^2 d\mathbf{r} = \frac{2\pi N_0 e^4}{T} r_\mathrm{D} = \frac{e^2}{4r_\mathrm{D}}$$

Equation (2.7) for the Debye-Hückel radius was used here. It can be readily seen that this equation is valid also for a plasma with different electron and ion temperatures.

The mean energy $\bar{\varepsilon}$ of a charged particle in a plasma is the sum of its mean kinetic energy $(3/2)T$ and the above energy of interaction between the charged particle and the plasma, that is

$$\bar{\varepsilon} = \frac{3}{2} T - \frac{e^2}{4r_\mathrm{D}}$$

Clearly, for an ideal plasma the second term in this equation is small in comparison with the first one.

Problem 2.14. Calculate the surface tension at the boundary of an ideal plasma.

The surface tension is the free energy of the unit surface area of plasma which is determined by the surface effect. Let us separate the plasma under consideration in two parts by an imaginary surface and move the parts away from each other. This changes the energies of both parts of plasma

The surface tension is equal to the variation of the energy of each part of plasma per unit surface area. We shall use this definition of the surface tension to calculate it.

We have to find the variation of the plasma energy as a result of the above procedure. As shown in Problem 2.13, the variation of the energy of a particle at a distance z from the surface, which is equal to the variation of the potential of its interaction with the plasma particles, is

$$\Delta U = \int_z^\infty \frac{e^2}{r} \exp(-r/r_D) \frac{e\varphi}{T} N_0 \, d\mathbf{r} = \int_z^\infty \frac{(e\varphi)^2}{T} N_0 \, d\mathbf{r}$$

where $\varphi = \dfrac{e}{r} \exp(-r/r_D)$ is the potential produced by the test particle in the plasma at a distance r from it. We assume here that the radius of curvature of the surface is large compared with the characteristic length of plasma, that is, the Debye-Hückel radius. Thus, a surface element can be regarded as being flat.

The variation of energy corresponding to the surface element dS in the above procedure is αdS, where α is the surface tension of plasma. On the other hand, microscopic analysis yields the following expression for this variation of energy:

$$2\int \Delta U(z) \, N_0 dS \, dz$$

where the z-axis is perpendicular to the surface, so that $N_0 dS \, dz$ is the number of charged particles in the given volume element, and the factor 2 takes into account that plasma contains two species of charged particles. Using the above equation for the variation of energy of a single charged particle, we can find the surface tension of plasma:

$$\alpha = 2N_0 \int_0^\infty \Delta U(z) \, dz = \frac{2e^2 N_0^2}{T} \int_0^\infty dz \int_z^\infty \varphi^2(r') \, d\mathbf{r}'$$

Changing the limits of integration, we obtain

$$\alpha = \frac{2e^2 N_0^2}{T} \int_{z>0} z' \varphi^2(r') \, d\mathbf{r}'$$

If we substitute the expression for the potential of a charged particle in a plasma and use the spherical coordinates, we obtain the following formula for the surface tension of the plasma:

$$\alpha = \frac{2e^2N_0^2}{T}\int_0^\infty r'^2\,dr'\int_0^1 2\pi\,d(\cos\theta')\cdot r'\cos\theta'\,\frac{e^2}{r'^2}\exp\left(-2r'/r_{\rm D}\right)$$

$$= \frac{\pi e^4N_0^2}{2T}r_{\rm D}^2 = \frac{e^2}{16}N_0$$

This result shows that the surface tension of the plasma has no practical significance. Indeed, if the radius of curvature of the surface is R, the pressure developed by the surface tension is of the order of α/R and its ratio to the plasma pressure is

$$\frac{\alpha}{RN_0T}\sim\frac{e^2}{TR}$$

and even if we assume that the radius of curvature of the plasma surface is of the order of the mean distance $N_0^{-1/3}$ between the charged particles (though we used the condition $R \gg r_{\rm D}$ above), we see that the above ratio for the ideal plasma is very small.

Problem 2.15. Determine the distribution function for the potentials acting on a charged test particle in a plasma.

We have to find the probability that the potential of interaction between a given charged particle and the surrounding charged particles has a given value. This potential should be expressed as the sum of the interaction potentials between the given particle and each of the surrounding particles. The potential of interaction between two charged particles in the plasma is $\frac{e^2}{r}\exp(-r/r_{\rm D})$, where r is the distance between the particles and $r_{\rm D}$ is the Debye-Hückel radius.

Let us estimate the characteristic potential acting on a test particle. We shall assume that this potential is produced by the particles in the region with the radius of the order of the Debye-Hückel radius around the test particle. The mean number of charged particles in this region $\bar{n}\sim Nr_{\rm D}^3$ (N is the number density of the plasma particles), and the potential of interaction of the test particle with each of these particles is of the order of $e^2/r_{\rm D}$. Fluctuations of the numbers of particles of both signs in this region are of the order of $\sqrt{\bar{n}}$ ($\bar{n}\gg 1$). Hence, the characteristic potential acting on the test particle is

$$\Delta U\sim\sqrt{\bar{n}}\,\frac{e^2}{r_{\rm D}}\sim e^2N^{1/2}r_{\rm D}^{1/2}$$

Compare this potential with the characteristic energy of a charged particle in the ideal plasma with the temperature T:

$$\frac{\Delta U}{T} \sim \left(\frac{Ne^6}{T^3}\right)^{1/4} \ll 1, \quad \frac{\Delta U}{e^2 N^{1/3}} \sim \left(\frac{T^3}{Ne^6}\right)^{1/12} \gg 1$$

These inequalities show that the characteristic potential acting on a charged particle in the ideal plasma is much lower than the thermal energy of particles and much higher than the energy of the pairwise interaction at the mean distance between the particles. Thus, each charged particle, on the one hand, can be regarded as being free and, on the other hand, the mean potential for the given particle is produced as a result of interactions with a large number of surrounding particles.

Therefore, we can write the potential distribution function $f(U)$ in the form of the normal distribution (1.88):

$$f(U)\,dU = (2\pi\Delta U^2)^{-1/2} \exp\,[(U - U)^2/2\Delta U^2]\,dU$$

Here $\Delta U^2 = \overline{U^2} - \overline{U}^2$, $\overline{U}$ is the mean potential of interaction between the test particle and the surrounding charged particles, and $\overline{U^2}$ is the mean square of this potential.

The potential $\overline{U}$ was found in Problem 2.13. It is determined by the variation of the density of the charged particles under the effect of the field of the test particle at distances of the order of the Debye-Hückel radius. This variation is small compared with the density itself and is of the order of $e^2/r_D T$ according to the Boltzmann distribution. Hence we can estimate $\overline{U}$:

$$\overline{U} \sim \frac{e^2}{r_D}\,\overline{n}\cdot\frac{e^2}{r_D T} \sim \frac{e^4}{T} N r_D \sim \frac{\Delta U^2}{T} \ll \Delta U$$

Since the mean potential acting on a test particle in a plasma is small compared with the characteristic potential, we shall assume that it is zero and, thus, we shall neglect the effect of the field of the test particle on the distribution of charged particles, since this effect is small. Then the positive and negative values of the total potential will have equal probabilities.

Using the equation for the potential of the pairwise interaction of two charged particles in a plasma, we obtain

$$\Delta U^2 = \overline{U^2} = 2\int_0^\infty \frac{e^4}{r^2} \exp\left(-\frac{2r}{r_D}\right) N_0 \cdot 4\pi r^2\,dr = 4\pi N_0 e^4 r_D$$

where the factor 2 takes into account the presence of the particles of opposite signs, and N_0 is the number density of the particles of one sign. This equation is identical to the above estimate. Substituting it into the normal distribution, we can find $f(U)\,dU$ (the probability that the potential acting on a charged test particle in the ideal plasma is in the range from U to $U+dU$):

$$f(U)\,dU = \frac{dU}{2\pi e^2 \sqrt{2N_0 r_D}} \exp\left[-\frac{U^2}{8\pi N e^4 r_D}\right]$$

2.3. MOTION OF ELECTRONS IN A GAS UNDER THE EFFECT OF EXTERNAL FIELDS

Motion of electrons in a weakly ionized gas determines its electric parameters and is, therefore, of practical interest. The behaviour of electrons in a gas depends on the effects of external fields and collisions between electrons and gas particles. Analyzing these factors, we can determine the energy distribution for electrons as illustrated in the problems discussed below. The small parameter making it possible to solve the Boltzmann kinetic equation for electrons is the ratio of the electron mass to the mass M of the gas particles. The approach to solving the kinetic equation can also make use of the fact that the cross section of elastic collisions between electrons and gas particles is always much greater than the cross section of inelastic scattering.

Problem 2.16. Determine the velocity distribution function for the electrons travelling in a uniform electric field if the collisions between electrons and gas atoms are elastic. The density of electrons is small, so that their collisions with each other can be ignored.

The kinetic equation for the velocity distribution function $f(\mathbf{v})$ of electrons in this case is

$$\frac{e\mathbf{E}}{m}\frac{\partial f}{\partial \mathbf{v}} = I_c(f) \tag{2.19}$$

where $\mathbf{v}$ is the electron velocity, m is the electron mass, $\mathbf{E}$ is the electric field strength, and I_c is the collision integral.

Since the electron mass m is much smaller than the mass M of the gas particle the energy exchanged in the collision between the electron and the gas particle is small [of the order of $(m/M)^{1/2}$ of the energy of these particles in centre-of-inertia reference frame]. This means that collision between the electron and the gas particle can sharply change the direction of the electron velocity, while the change in the velocity magnitude is small. Therefore, the velocity distribution function of the electrons shows no significant dependence on velocity direction, that is, it can be regarded as being close to spherically symmetric. The deviation from spherical symmetry of the distribution is determined by the external electric field. Since such deviation is small, the velocity distribution function of the electrons can be written as

$$f(\mathbf{v}) = f_0(v) + v_x f_1(v) \tag{2.20}$$

where the x-axis is parallel to the external electric field.

The collision integral describing the collisions between the electrons and the gas particles is a linear function with respect to the distribution function: $I_c(f) = I_c(f_0) + I_c(v_x f_1)$. The collision integral for the spherically symmetric part of the velocity distribution was derived above; for elastic collisions between the electrons and the gas particles it is given by eq. (1.74):

$$I_c(f_0) = \frac{1}{v^2}\frac{\partial}{\partial v}\left[\frac{m}{M} T v^3 \nu_{el}\left(\frac{f_0}{T} + \frac{\partial f_0}{mv\,\partial v}\right)\right] \tag{2.21}$$

Here $\nu_{el} = Nv\sigma^*$ is the frequency of elastic collisions between the electrons and the gas particles. The other part of the collision integral is

$$I_c(v_x f_1) = \int N_a[f_a(\mathbf{v}_a') f_1(v') v_x' - f_a(\mathbf{v}_a) f_1(v) v_x] |\mathbf{v} - \mathbf{v}_a| \, d\sigma \, d\mathbf{v}_a$$

where $\mathbf{v}_a$ and $\mathbf{v}_a'$ and $\mathbf{v}$ and $\mathbf{v}'$ are the velocities of the gas particle and electron before and after the collision, f_a is the distribution function for the gas particles normalized to unity $\left(\int f_a d\mathbf{v}_a = 1\right)$, N_a is the density of the gas particles, and $d\sigma$ is the differential cross section for scattering of electrons by the gas particles. Let us make use of the fact that the variations of the energy of the electron and the velocity of the gas particle in collision are small ($\mathbf{v}_a = \mathbf{v}_a'$, $\mathbf{v} = \mathbf{v}'$) and that $v \gg v_a$. We obtain

$$\begin{aligned} I_c(v_x f_1) &= - N_a \int f_a(\mathbf{v}_a) \, d\mathbf{v}_a f_1(v) (\mathbf{v} - \mathbf{v}')_x v \, d\sigma \\ &= - N_a v f_1(v) v_x \sigma^* = - \nu_{el} v_x f_1(v) \end{aligned} \tag{2.22}$$

Substitute eqs. (2.20), (2.21) and (2.22) into the kinetic equation (2.19) and integrate the result over $\cos\vartheta$ (ϑ is the angle between $\mathbf{v}$ and $\mathbf{E}$). We obtain

$$\frac{eE}{m}\left(\frac{v}{3}\frac{\partial f_1}{\partial v} + f_1\right) \equiv \frac{eE}{3mv^2}\frac{\partial f_1 v^3}{\partial v} = I_c(f_0) \tag{2.23a}$$

Another equation for f_0 and f_1 is derived by multiplying eq. (2.19) by $\cos\vartheta$ and integrating the result over $d(\cos\vartheta)$:

$$\frac{eE}{m}\frac{\partial f_0}{\partial v} = - \nu_{el} v f_1. \tag{2.23b}$$

Using eq. (1.74) for the collision integral $I_c(f_0)$, we obtain the following solution of the above system of differential equations in f_0 and f_1:

$$f_0 = C \exp\left[-\int_0^v \frac{mv\,dv}{\left(T + \dfrac{e^2E^2M}{3m^2\nu_{el}^2}\right)} \right] \tag{2.24a}$$

$$f_1 = -\frac{eE}{mv\nu_{el}}\frac{df_0}{dv} = \frac{eE}{\nu_{el}}\frac{f_0}{\left(T + \dfrac{e^2E^2M}{3m^2\nu_{el}^2}\right)} \tag{2.24b}$$

The constant C can be found from the normalization condition $\int f(\mathbf{v})\,d\mathbf{v} = \int f_0(\mathbf{v})\,d\mathbf{v} = 1$.

Let us estimate the ratio vf_1/f_0. It has the maximum value of $eE/m\nu \sim$ $\sim (T/M)^{1/2}$, which is of the order of $(m/M)^{1/2}$ $(v \sim \sqrt{T/m})$. This means that vf_1/f_0 is always much smaller than unity. Hence, our assumption that the distribution function is close to being spherically symmetric is valid.

The velocity distribution function for the electrons derived from eqs. (2.24) for strong external fields is known as the Druyvesteyn distribution. (Originally, Druyvesteyn analyzed the case $\sigma = \text{const}$, $E \to \infty$.)

Problem 2.17. Find the drift velocity and the mean energy of electrons at strong electric fields under the conditions of Problem 2.16. Assume that the dependence of the collision frequency for electrons and gas atoms on the collision velocity is of the form $\nu = Cv^n$, where $n > -1$.

As shown in Problem 2.16, the spherically symmetric part of the velocity distribution function for electrons for strong external electric fields is

$$f_0 = \text{const} \cdot \exp\left[-\frac{3mv^{2n+2}C^2}{2(n+1)Ma^2} \right]$$

where $a = eE/m$, and the collision frequency is $\nu = Cv^n$. Hence the mean electron energy is

$$\bar{\varepsilon} = \left\langle \frac{mv^2}{2} \right\rangle = \frac{m}{2}\left[\frac{2Ma^2(n+1)}{3mC^2}\right]^{\frac{1}{n+1}} \frac{\Gamma\left(\dfrac{5}{2n+2}\right)}{\Gamma\left(\dfrac{3}{2n+2}\right)}$$

The electron drift velocity is $w = \langle v_x \rangle = \left\langle v_x^2 \dfrac{f_1}{f_0} \right\rangle$. In Problem

2.16 we have found that $f_1(v) = \frac{m}{M}\cdot\frac{3m\nu}{eE}f_0$ and hence

$$w = \frac{m}{Ma}\langle v^2\nu\rangle = \frac{mC}{Ma}\left[\frac{2Ma^2(n+1)}{3mC^2}\right]^{\frac{2+n}{2n+2}} \frac{\Gamma\left(\frac{5+n}{2n+2}\right)}{\Gamma\left(\frac{3}{2n+2}\right)}$$

For instance, if the collision frequency is independent of the collision velocity ($n = 0,\ \nu = C$), we have

$$\bar{\varepsilon} = \frac{Ma^2}{2\nu^2} = \frac{Me^2E^2}{2m^2\nu^2}, \quad w = \frac{eE}{m\nu} \tag{2.25a}$$

that is, $\bar{\varepsilon} = Mw^2/2$.

In another limiting case, when the collision cross section for electrons and atoms is independent of velocity, we have $n = 1$, $C = 1/\lambda$ (λ is the electron's mean free path) and

$$\begin{aligned} \bar{\varepsilon} &= \frac{m}{2}\left(\frac{4Ma^2\lambda^2}{3m}\right)^{1/2}\frac{\Gamma(5/4)}{\Gamma(3/4)} = 0.427\sqrt{\frac{M}{m}}\,eE\lambda \\ w &= \frac{m}{Ma\lambda}\left(\frac{4Ma^2\lambda^2}{3m}\right)^{3/4}\frac{\Gamma(3/2)}{\Gamma(3/4)} = 0.897\,\frac{\sqrt{eE\lambda}}{M^{1/4}m^{1/4}} \end{aligned} \tag{2.25b}$$

that is, $\bar{\varepsilon} = 0.530Mw^2$.

> **Problem 2.18.** Find the velocity distribution function for the electrons in a molecular gas to which a uniform electric field is applied. Electrons lose energy in elastic collisions with molecules, and excitations of the first vibrational levels of molecules. The excitation energy for the vibrational levels of molecules is much smaller than the mean electron energy and the electron temperature is much higher than the gas temperature.

The cross section of inelastic scattering of electrons by the gas molecules is always much smaller than the cross section of elastic scattering. Therefore, collisions between electrons and the gas molecules result in greater variation of the direction of the electron velocity than is the case for the electron energy and we can make use of eq. (2.20) for the velocity distribution function for electrons. Equation (2.23b) remains unchanged and in eq. (2.23a) the collision integral $I_c(f_0)$ is determined by the excitation of molecular vibrational levels and elastic scattering of electrons by the molecules. According to eqs. (1.74) and (1.75) for high electron energies, we have

$$I_c(f_0) = \frac{1}{v^2}\frac{\partial}{\partial v}\left(\frac{m}{M}v^3\nu_{el}f_0 + \frac{\hbar\omega}{m}v\nu_{exc}f_0\right)$$

Using this expression for solving the system of equations (2.23), we find the spherically symmetric part of the distribution function:

$$f_0(v) = C\exp\left[-\int_0^v \frac{mv\,dv}{\frac{M}{3}\left(\frac{eE}{m\nu_{el}}\right)^2}\left(1+\frac{M}{m}\frac{\nu_{exc}}{\nu_{el}}\frac{\hbar\omega}{mv^2}\right)\right] \qquad (2.26)$$

If we ignore the excitation of the vibrational levels ($\nu_{exc} \to 0$), eq. (2.26) is identical to the formula derived from eq. (2.24a) for electron energies much higher than the thermal energy of the gas particles.

Problem 2.19. Derive the integral equations for the mean momentum and mean energy for electrons moving in a molecular gas to which a uniform electric field is applied. The rate constant for elastic collisions of electrons with molecules is much higher than the rate constant of excitation of the vibrational molecular levels. The loss of electron energy is due to the excitation of vibrational levels while de-excitation of molecules does not depend on collisions with the electrons.

Under the conditions of the problem the variation of the electron momentum is determined by elastic collisions with the molecules and the equation for the variation of the mean momentum has the same form as in Problem 1.10 [see eq. (1.22)]:

$$eE = mN\langle v_x v\sigma^*(v)\rangle$$

Here E is the external electric field, m is the electron mass, N is the number density of molecules, v is the electron velocity equal here to the relative velocity of a collision between an electron and a molecule, and σ^* is the diffusion cross section of scattering of the electron by the molecule. In particular, if the frequency of collisions between electrons and molecules is independent of the electron velocity ($\nu = Nv\sigma^*(v) = \text{const}$), the mean electron velocity is

$$w = eE/m\nu$$

The equation for the mean energy is determined here by the excitation of the vibrational molecular levels. If we take into account only the vibrational transitions in molecule due to collisions with electrons, this equation has the following general form:

$$eEw = \sum_{i,n} [\hbar\omega(n-i)\,k_{in}N_i - \hbar\omega(n-i)\,k_{ni}N_n]$$

The left-hand side of the equation is the energy acquired by the electron from the field and the right-hand side is the energy spent for the excitation of the vibrational levels. Here N_i is the number density of molecules in the ith vibrational state, $\hbar\omega$ is the energy difference between the neighbouring vibrational states, and $k_{in} = \langle v\sigma_{in}\rangle$ is the rate constant for the transition from the ith vibrational state to the nth vibration state owing to a collision with an electron.

If the number of molecules in the excited vibrational states is small and the molecules are effectively excited to the first vibrational state, the balance equation for the mean electron energy can be modified into

$$eEw = \hbar\omega N k_{01}$$

The balance equations for the mean momentum and mean energy of electrons, generally speaking, make it possible to find the mean electron energy as a function of the parameter eE/N.

Problem 2.20. Find the energy distribution function for electrons in a gas in a uniform electric field taking into account the excitation of the electronic states of the gas atoms. Assume that the mean electron energy is much lower than the energy of excitation of the gas atoms.

In this case the number of fast electrons capable of exciting or ionizing the gas atoms is small, so that the loss of energy by these electrons due to the excitation of atoms will not change the energy distribution of slow electrons. The collision integral for the spherically symmetric part of the distribution function can be written as $I_c(f_0) = -\nu_{in} f_0$, where $\nu_{in} = N_a v\sigma_{in}(v)$ is the frequency of inelastic collisions between electrons and atoms, N_a is the density of atoms, v is the electron velocity, and σ_{in} is the total cross section of excitation and ionization of atoms by an electron impact. We assume that the electron density is small, so that an excited atom has time for de-excitation to the ground state or leaving the system before the next collision with an electron can occur. The equation for the spherically symmetric part of the distribution function is

$$\frac{e^2E^2}{3m^2\nu_{in}v^2}\frac{d}{dv}\left(\frac{v^2}{\nu_{el}}\frac{df_0}{dv}\right) - f_0 = 0$$

Let us make use of the assumption that the mean electron energy is small compared with the excitation energy. Applying the quasiclassical approach, assume that $f_0 = \exp(S)$, $v\,dS/dv \gg 1$ and S is a smoothly varying function of the velocity [that is, $(S')^2 v \gg |S''|$]. We obtain

$$f_0(v) = A\exp\left[-\int_{v_0}^{v}\sqrt{3}\,\frac{\nu m}{eE}\,dv\right] \qquad (2.27)$$

where $\nu = \sqrt{\nu_{el}\nu_{in}}$, $mv_0^2/2$ is the excitation energy for atoms, and the factor A is found by joining the distribution functions (2.24) and (2.27) at the energies when $\nu_{in} \sim \frac{m}{M}\nu_{el}$. These energies are near the excitation threshold, since the characteristic excitation frequencies practically always satisfy the following inequalities:

$$\frac{m}{M}\nu_{\mathrm{el}} \ll \nu_{\mathrm{in}} \ll \nu_{\mathrm{el}}$$

If this frequency condition is satisfied, the distribution functions (2.24) and (2.27) can be joined practically at the threshold.

Equation (2.27) is valid when the condition $vS' \gg 1$ of applicability of the quasiclassical approximation is satisfied, that is, when

$$eE/v_0 m \ll \sqrt{\nu_{\mathrm{el}}\nu_{\mathrm{in}}}$$

Problem 2.21. A uniform electric field is applied to a gas. Estimate the time in which the equilibrium drift velocity and the equilibrium mean energy of electrons are reached.

Multiply the kinetic equation

$$\frac{\partial f}{\partial t}+\frac{e\mathbf{E}}{m}\frac{\partial f}{\partial \mathbf{v}} = I_{\mathrm{c}}(f)$$

by $\mathbf{v}$ and integrate the result over the electron velocity. We obtain

$$\left\langle \mathbf{v}\frac{\partial f}{\partial t}\right\rangle = \frac{\partial}{\partial t}\overline{\langle \mathbf{v}\rangle} = \frac{d\mathbf{w}}{dt}$$

$$\frac{eE}{m}\int \mathbf{v}\frac{\partial f}{\partial \mathbf{v}}\,d\mathbf{v} = -\frac{e\mathbf{E}}{m}\int f\,d\mathbf{v} = -\frac{e\mathbf{E}}{m}$$

where $\mathbf{w}$ is the electron drift velocity. Hence we have

$$\frac{d\mathbf{w}}{dt} = \frac{e\mathbf{E}}{m}+\int I_{\mathrm{c}}(f)\,\mathbf{v}\,d\mathbf{v}$$

Since the frequency of elastic collisions between electrons and gas particles is higher than the frequencies of other collision processes, we obtain

$$I_{\mathrm{c}}(f) = I_{\mathrm{c}}(f_0) - \nu_{\mathrm{el}}v_x f_1(v)$$

$$\int I_{\mathrm{c}}\mathbf{v}\,d\mathbf{v} = -\frac{\mathbf{n}}{3}\int \nu_{\mathrm{el}}v^2 f_1\,d\mathbf{v}$$

where $\mathbf{n}$ is the unit vector parallel to $\mathbf{E}$. Hence,

$$\frac{dw}{dt} = \frac{eE}{m}-\frac{1}{3}\int \nu_{\mathrm{el}}\,v^2 f_1\,d\mathbf{v}$$

Since the drift velocity $w = \int v_x f\,d\mathbf{v} = \frac{1}{3}\int v^2 f_1\,d\mathbf{v}$, the last term in the above equation is of the order of $\nu_{\mathrm{el}}w$. Hence, the equilibrium drift velocity is of the order of $eE/m\nu_{el}$ and the time needed to reach it is of the order of $1/\nu_{el}$.

The equation for the mean electron energy $\bar{\varepsilon}$ can be derived by multiplying the kinetic equation by $mv^2/2$ and integrating the result over the electron velocity. We obtain

$$\int \frac{mv^2}{2} \frac{\partial f}{\partial t} d\mathbf{v} = \frac{\partial}{\partial t} \int \frac{mv^2}{2} f \, d\mathbf{v} = \frac{d\bar{\varepsilon}}{dt}$$

$$\int \frac{mv^2}{2} \frac{e\mathbf{E}}{m} \frac{\partial f}{\partial \mathbf{v}} d\mathbf{v} = -\frac{e\mathbf{E}}{m} \int m\mathbf{v} f \, d\mathbf{v} = -e\mathbf{E}\mathbf{w}$$

Finally we have

$$\frac{d\bar{\varepsilon}}{dt} = eEw + \int \frac{mv^2}{2} I_c(f_0) \, d\mathbf{v} \, .$$

$$\int \frac{mv^2}{2} I_c(f_0) \, d\mathbf{v}$$

$$= \int \frac{mv^2}{2} \frac{1}{v^2} \frac{\partial}{\partial v} \left[v^3 \nu_{el} \frac{m}{M} T \left(\frac{f_0}{T} + \frac{\partial f_0}{mv \, \partial v} \right) \right] 4\pi v^2 \, d\mathbf{v}$$

$$= -\frac{m^2}{M} T \int v^2 \nu_{el} \left(\frac{f_0}{T} + \frac{\partial f_0}{mv \, \partial v} \right) d\mathbf{v}$$

It can be seen that the last term is of the order of $\frac{m}{M} \nu_{el} \bar{\varepsilon}$, that is, the time needed to establish the equilibrium electron energy is of the order of $M\nu_{el}/m$, which is much longer than the time needed to establish the equilibrium drift velocity. Therefore, when an electron travels in a gas, it has the drift velocity $w(v)$ corresponding to a given electron energy. The drift velocity of translational motion at each moment is uniquely determined by the electron energy and is $\frac{1}{3} v \frac{vf_1}{f_0}$, that is, it is considerably lower than the electron velocity.

This situation prevails also if the electron energy losses are determined by inelastic collisions between electrons and gas particles. Under such conditions the frequency of reaching the equilibrium electron energy is of the order of, or lower than, the frequency ν_{in} of inelastic collisions between electrons and gas particles, which is always considerably lower than the frequency of elastic collisions. Therefore, the velocity of translational electron motion at any moment is determined by the electron energy.

Problem 2.22. Find the drift velocity and the mean energy of electrons in the case of elastic collisions between electrons and gas atoms if the time of energy transfer in collisions of two electrons is shorter than in collisions between electrons and gas atoms. Analyze two cases: $\nu_{el} = \text{const}$, and $\nu_{el} = cv$.

We have the following system of equations for f_0 and f_1:

$$\frac{eE}{3m}\frac{df_1v^3}{v^2\,dv} = I_{ea} + I_{ee}, \qquad \frac{eE}{m}\frac{df_0}{dv} = -\nu_{el}vf_1$$

Here I_{ee} is the integral of interelectron collisions, and I_{ea} is the integral of collisions between electrons and the gas atoms. The condition of this problem is that $\nu_{ee} \sim I_{ee} \gg I_{ea} \sim \frac{m}{M}\nu_{ea}$. Multiply the first equation by $mv^2/2$ and integrate the result over the electron velocities using the fact that, according to eq. (1.15),

$$\int I_{ee}v^2\,d\mathbf{v} = 0$$

We obtain

$$eEw = \int \frac{mv^2}{2} I_{ea}\,d\mathbf{v}$$

where w is the electron drift velocity. Now we see that the left-hand side of the first equation is of the order of I_{ea}, that is, it is much smaller than I_{ee}. Therefore, in the zero approximation this equation yields $I_{ee} = 0$. A solution of this equation is the Maxwellian distribution

$$f_0 = \left(\frac{m}{2\pi T_e}\right)^{3/2} \exp\left(-\frac{mv^2}{2T_e}\right)$$

Hence we obtain $f_1 = eEf_0/\nu_{el}T_e$, so that

$$w = \frac{eE}{T_e}\left\langle\frac{v^2}{3\nu_{el}}\right\rangle$$

In our problem, when $\nu_{el} = \nu_T\left(\frac{mv^2}{2T_e}\right)^{n/2}$, we have

$$w = \frac{2eE}{3mv_T}\int_0^\infty \frac{4}{\sqrt{\pi}} \exp(-x^2)\,x^{4-n}\,dx = \frac{8}{3\sqrt{\pi}}\frac{eE}{mv_T}\Gamma\left(\frac{5-n}{2}\right)$$

where $x = \sqrt{mv^2/2T_e}$. Furthermore, we have

$$-\left\langle \frac{mv^2}{2} I_{ea} \right\rangle = -T_e\langle x^2 I_{ea}\rangle = +2T_e \int x\,dx \frac{m}{M} Tv^3\nu_{el}\left(\frac{f_0}{T} + \frac{\partial f_0}{mv\,dv}\right)$$

$$= 2T_eT\left(\frac{1}{T} - \frac{1}{T_e}\right)\frac{m}{M}\langle x^2\nu_{el}\rangle = \frac{8}{\sqrt{\pi}}\frac{m}{M}(T_e - T)\,\nu_T\Gamma\left(\frac{5+n}{2}\right)$$

Since $eEw = -\left\langle \frac{mv^2}{2} I_{ea} \right\rangle$, we obtain

$$T_e - T = M\frac{e^2E^2}{3m^2\nu_T^2}\frac{\Gamma[(5-n)/2]}{\Gamma[(5+n)/2]}$$

for $n = 0$

$$T_e - T = M\frac{e^2E^2}{3m^2\nu_T^2}$$

for $n = 1$

$$T_e - T = M\frac{e^2E^2}{6m^2\nu_T^2}$$

Let us derive these estimates for strong fields from simple physical considerations. The variation of the electron velocity in the period between two successive collisions with the gas atoms is $\Delta v \sim eE/m\nu$. The velocity distribution of electrons is spherically symmetric, so that the mean variation of the electron energy between two successive collisions owing to the effect of the external field $m\Delta v^2$ is of the order of $m(eE/m\nu)^2$. This energy is transferred to the gas atoms, and the energy transferred to an atom in each collision is of the order of $m\bar{\varepsilon}/M$, where $\bar{\varepsilon}$ is the mean electron energy.

Hence, the mean electron energy is

$$\bar{\varepsilon} \sim M\left(\frac{eE}{m\nu}\right)^2$$

for strong external electric fields, when the mean electron energy is much higher than the thermal energy of the gas particles.

Problem 2.23. Estimate the mean energy of electrons in a monoatomic gas to which a variable electric field and a constant magnetic field perpendicular to the electric field are applied. Assume that the mean electron energy is much higher than the thermal energy of the gas atoms and ignore the inelastic collision between electrons and gas atoms.

Let us estimate, at first, the mean energy of an electron moving in a gas to which a uniform electric field is applied. The energy acquired by the electron per unit time from the electric field must be equal to the energy lost by the electron per unit time in binary collisions with the gas atoms. The mean energy lost by the electron and acquired by the gas atom in the collision between them is

$$\Delta\varepsilon = \frac{(\mathbf{P} + \Delta\mathbf{P})^2}{2M} - \frac{P^2}{2M} = \frac{\mathbf{P}\Delta\mathbf{P}}{M} + \frac{\Delta P^2}{2M} = -\frac{\mathbf{P}\Delta\mathbf{P}}{M} + \frac{\Delta p^2}{2M}$$

Here M is the mass of the gas atom, $\mathbf{P}$ is the momentum of the atom before collision, $\Delta\mathbf{P}$ is the variation of the atom's momentum in collision, and $\Delta\mathbf{p}$ is the variation of the electron's momentum (according to the law of conservation of momentum $\Delta\mathbf{P} = -\Delta\mathbf{p}$).

Since the atomic mass is much larger than the electron mass, we can assume that the atom is infinitely heavy and resting. Therefore, the electron's momentum $\mathbf{p}$ and its variation $\Delta\mathbf{p}$ in collision with the atom are independent of the momentum $\mathbf{P}$ of the atom. Averaging the above equation for $\Delta\varepsilon$ over the directions of the electron's momentum, we obtain $\Delta\varepsilon = \Delta p^2/2M$. Since $\Delta p \sim p$ and $p \sim \sqrt{m\bar{\varepsilon}}$ (where $\bar{\varepsilon}$ is the mean electron energy), we find that the energy transferred from the electron to the atom during one collision is $\Delta\varepsilon \sim \frac{m}{M}\bar{\varepsilon}$ and the energy transferred per unit time is $\Delta\varepsilon\nu \sim \nu\frac{m}{M}\bar{\varepsilon}$ (here $\nu = N\langle v\sigma_{\text{ea}}\rangle$ is the frequency of elastic collisions between electrons and the gas atoms, N is the number density of atoms, σ_{ea} is the cross section of elastic collision between an electron and an atom, and angle brackets denote averaging over the electron velocities v).

The energy acquired by the electron from the electric field per unit time is $e\mathbf{E}\mathbf{w}$, where $w = eE/m\nu$ is the velocity of the translational electron motion. Writing the balance equation for the variation of the electron energy per unit time, we find the following estimate of the mean electron energy, which is identical to the result obtained in Problem 2.22:

$$\bar{\varepsilon} \sim M\left(\frac{eE}{m\nu}\right)^2$$

This estimate is valid if the mean electron energy is high compared with the thermal energy of the gas atoms (that is, if collisions with the gas atoms result only in losses of electron energy), and small compared with the characteristic electron energies at which the inelastic scattering processes become significant.

A similar approach can be used to find the mean energy of electrons in crossed constant electric and magnetic fields. According to the results of Problem 2.35, the energy acquired by the electron from the field is $eEw = (eE)^2\tau/m(1 + \omega_0^2\tau^2)$, so that the mean electron energy

$$\bar{\varepsilon} \sim M\left(\frac{eE}{m}\right)^2(\nu^2 + \omega_0^2)^{-1}$$

Let us, finally, analyze the motion of an electron in a gas in a varying electric field. If the electric field varies as $E\cos\omega t$, then, according to the results of Problem 2.39, we can take the real part of the expression for the drift velocity of the electron along the field and obtain

$$w_x = \frac{eE}{m}(\omega^2 + \nu^2)^{-1}(\nu\cos\omega t + \omega\sin\omega t)$$

The energy acquired by the electron per unit time averaged over time is

$$\overline{eEw_x} = \frac{(eE)^2\nu}{2m(\omega^2 + \nu^2)}$$

Hence the mean energy of the electron in a gas in a varying electric field is

$$\bar{\varepsilon} \sim M\left(\frac{eE}{m}\right)^2(\nu^2 + \omega^2)^{-1}$$

Problem 2.24. Find the distribution function for the electrons in a gas to which crossed electric and magnetic fields are applied. Assume that the energy loss of electrons is due to elastic collisions between electrons and atoms.

The kinetic equation for the velocity distribution of electrons is

$$\left(e\mathbf{E} + \frac{e}{c}[\mathbf{vH}]\right)\frac{\partial f}{m\,\partial\mathbf{v}} = I_c(f)$$

where **E** and **H** are the strengths of the electric and magnetic fields, respectively, and I_c is the collision integral. Let **E** be parallel to the x-axis and **H** to the z-axis. Rewrite the kinetic equation in the following form:

$$(a + \omega_0 v_y)\frac{\partial f}{\partial v_x} - v_x\omega_0\frac{\partial f}{\partial v_y} = I_c(f)$$

Here $a = eE/m$, and $\omega_0 = eH/mc$ is the Larmor frequency. The distribution function for electrons is close to being spherically symmetric; the electric field causes the directed motion of electrons along the x-axis and the magnetic field can cause the directed motion of electrons along the x- and y-axes. We can express the distribution function as

$$f = f_0(v) + v_x f_1(v) + v_y f_2(v)$$

The integral of collisions between electrons and atoms can be expressed as

$$I_c(f) = I_c(f_0) - \nu v_x f_1 - \nu v_y f_2$$

where $\nu = N_a v\sigma^*$ is the frequency of elastic collisions and

$$I_c(f_0) = \frac{m}{M} T \frac{1}{v^2} \frac{\partial}{\partial v} \left[v^3 \nu \left(\frac{f_0}{T} + \frac{\partial f_0}{mv\, \partial v} \right) \right]$$

Then the kinetic equation can be rewritten as

$$a \frac{v_x}{v} \frac{df_0}{dv} + a \left(f_1 + \frac{v_x^2}{v} \frac{df_1}{dv} \right) + a \frac{v_x v_y}{v} \frac{df_2}{dv} + \omega_0 v_y f_1 - \omega_0 v_x f_2$$

$$= I_c(f_0) - \nu v_x f_1 - \nu v_y f_2$$

Average this equation over the angles of the vector **v**, multiply it by v_x/v and v_y/v and average the results over the angles. We obtain the following system of equations for the functions f_0, f_1 and f_2:

$$\begin{gathered} \frac{a}{3} \frac{df_1 v^3}{v^2\, dv} = \frac{m}{M} T \frac{1}{v^2} \frac{d}{dv} \left[v^3 \nu \left(\frac{f_0}{T} + \frac{df_0}{mv\, dv} \right) \right] \\ a \frac{df_0}{dv} - \omega_0 v f_2 = -\nu v f_1 \\ \omega_0 f_1 = -\nu f_2 \end{gathered} \tag{2.28}$$

A solution of this system of equations is

$$f_0 = C \exp \left\{ - \int_0^v \frac{mv\, dv}{\left[T + \dfrac{e^2 E^2 M}{3m^2(\nu^2 + \omega_0^2)} \right]} \right\}$$

$$f_1 = - \frac{eE\nu}{(\nu^2 + \omega_0^2)} \frac{df_0}{mv\, dv} = \frac{eE\nu}{(\nu^2 + \omega_0^2)} \left[T + \frac{e^2 E^2 M}{3m^2(\nu^2 + \omega_0^2)} \right]^{-1} f_0$$

$$f_2 = - \frac{\omega_0}{\nu} f_1 = - \frac{eE\omega_0}{(\nu^2 + \omega_0^2)} \left[T + \frac{e^2 E^2 M}{3m^2(\nu^2 + \omega_0^2)} \right]^{-1} f_0$$

where C is a normalization constant.

Problem 2.25. Find the drift velocity of electrons in a gas in crossed electric and magnetic fields and the conductivity of a weakly ionized plasma in crossed electric and magnetic fields.

The appropriate distribution function for electrons has been derived in Problem 2.24, where it was assumed that the electron energy is changed as a result of elastic collisions with the gas atoms. In the present problem this assumption is not made but the spherically symmetric part of the distribution function is assumed to be known. Since the cross section for elastic collisions between an electron and an atom is considerably higher than the cross section for inelastic collisions, the two last equations of the system (2.28) remain valid in this case. These equations can be used to find the drift velocity from the spherically symmetric part of the distribution function. If the electric field **E** is parallel to the x-axis and the magnetic field **H** parallel to the y-axis, the components of the conductivity tensor can be expressed in terms of the drift velocity as follows:

$$\Sigma_{xx} = \frac{N_e e w_x}{E}, \quad \Sigma_{yx} = \frac{N_e e w_y}{E}$$

Here w_x and w_y are the components of the drift velocity of electrons, and N_e is the electron number density.

The second and third equations of the system (2.28) yield

$$v f_1 = -\frac{a v}{(v^2 + \omega_0^2)} \frac{df_0}{dv}, \quad v f_2 = \frac{a \omega_0}{(v^2 + \omega_0^2)} \frac{df_0}{dv} \qquad \left(a = \frac{eE}{m}\right)$$

Hence the drift velocity components for electrons are

$$w_x = \frac{eE}{3m} \left\langle \frac{1}{v^2} \frac{d}{dv} \left(\frac{\nu v^3}{\nu^2 + \omega_0^2} \right) \right\rangle$$

$$w_y = -\frac{eE\omega_0}{3m} \left\langle \frac{1}{v^2} \frac{d}{dv} \left(\frac{v^3}{\omega_0^2 + \nu^2} \right) \right\rangle \tag{2.28 a}$$

Here the angle brackets denote the averaging over the spherically symmetric part of the distribution function: $\langle X \rangle \equiv \int X f_0 \, d\mathbf{v}$.

If the magnetic field strength is high ($\nu/\omega_0 \ll 1$), electrons drift perpendicular to the electric and magnetic fields ($w_y \gg w_x$). The above equations yield the following expression for the drift velocity:

$$w_y = -\frac{eE}{m\omega_0}$$

The drift velocity is independent of the frequency of collisions between electrons and atoms. This expression can be derived directly from the equation of motion of the electron if its collisions with atoms are ignored:

$$m \frac{d\mathbf{w}}{dt} = e\mathbf{E} + \frac{e}{c}[\mathbf{w}\mathbf{H}]$$

The solution of this equation for the stationary case ($d\mathbf{w}/dt = 0$) is

$$w_y = -\frac{cE}{H} = -\frac{eE}{m\omega_0}$$

Problem 2.26. A weakly ionized monoatomic gas flows with the velocity u parallel to two short-circuited electrodes. An external magnetic field H is perpendicular to the gas flow and parallel to the electrodes. An electron current flows between the electrodes under the effect of the magnetic field, and the electron temperature increases (the system is a short-circuited magnetohydrodynamic duct with continuous electrodes). Determine the maximum possible temperature difference between electrons and the gas.

In the stationary frame of reference the magnetic field gives rise to the electric field $E' = \frac{u}{c} H$. This field gives rise to the current flowing between the electrodes and the current flowing along the gas flow and closing at the electrodes. Let us write down the balance equation for the electron energy. The energy acquired by an electron from the field per unit time is $eE'w$, where w is the drift velocity of the electron along the induced electric field. This energy is transferred from the electron to the gas atoms in elastic collisions; as calculated in Problem 2.22, it equals $\frac{m}{M}(T_e - T)\left\langle \frac{mv^2}{T_e} \nu_{el} \right\rangle$, where m and M are the masses of the electron and the gas atom, ν_{el} is the frequency of elastic collisions between electrons and atoms, and the angle brackets denote averaging over the Maxwellian distribution function for electrons. Using the general expression for the drift velocity of electrons derived in Problem 2.25, which in our case be rewritten as $w = \frac{a}{3}\left\langle \frac{mv^2}{T_e} \frac{\nu_{el}}{\omega_0^2 + \nu_{el}^2} \right\rangle$, we obtain from the balance equation for the electron energy

$$(T_e - T)\frac{m}{M}\left\langle \frac{mv^2}{T_e} \nu_{el} \right\rangle = \frac{ma^2}{3}\left\langle \frac{mv^2}{T_e} \frac{\nu_{el}}{\omega_0^2 + \nu_{el}^2} \right\rangle$$

Since $a = eE'/m = eHu/mc = \omega_0 u$, this yields

$$T_e - T = \frac{Mu^2}{3} \frac{\left\langle v^2\omega_0^2 \frac{\nu_{el}}{(\omega_0^2 + \nu_{el}^2)} \right\rangle}{\langle v^2 \nu_{el} \rangle} \tag{2.29}$$

The difference $T_e - T$ is maximum for $\nu_{el}/\omega_0 \to 0$. For small ratios ν_{el}/ω_0 the temperature difference is

$$T_e - T = Mu^2/3$$

Problem 2.27. Calculate the heat conductivity due to the motion of electrons in a weakly ionized plasma. Use the Maxwellian distribution of electrons and assume that the electron temperature T_e is much higher than the gas temperature T and the frequency of elastic collisions between electrons and the gas atoms varies as $\nu_{el} \sim v^{-n}$, where v is the electron velocity.

Write the velocity distribution function for electrons as $f = f_0(v) + \mathbf{v}\nabla \ln T_e f_1(v)$, where $f_0 = (m/2\pi T_e)^{3/2} \exp(-mv^2/2T_e) N_e$ and T_e is the electron temperature. The kinetic equation in this case can be written in the form

$$f_0\left(\frac{mv^2}{2T_e} - \frac{5}{2}\right)\mathbf{v}\nabla \ln T_e = I_c(f)$$

Assume that the electron temperature variation at distances of the order of the electron mean free path is small. Therefore, a temperature gradient does not change the velocity distribution of electrons. Since $I_c(f_0) = 0$ and $I_c[v_x f_1(v)] = -\nu_{el} v_x f_1(v)$, we obtain

$$f_1(v) = -\frac{1}{\nu_{el}}\left(\frac{mv^2}{2T_e} - \frac{5}{2}\right) f_0(v)$$

where $\nu_{el} = N_a v \sigma^*$ and σ^* is the diffusion cross section of elastic collisions between electrons and atoms.

The heat flux is

$$q_i = \int \frac{mv^2}{2} v_i f \, d\mathbf{v} = -\int \frac{mv^2}{2} v_i v_k \frac{\partial \ln T_e}{\partial x_k} \frac{1}{\nu_{el}} \left(\frac{mv^2}{2T_e} - \frac{5}{2}\right) f_0 \, d\mathbf{v}$$

$$= -\frac{1}{3}\frac{\partial T_e}{\partial x_i} \int \frac{mv^2}{2T_e} v^2 \left(\frac{mv^2}{2T_e} - \frac{5}{2}\right) \frac{f_0}{\nu_{el}} d\mathbf{v} = -\varkappa \frac{\partial T}{\partial x_i}$$

According to the results of Problem 2.22, for a given dependence of the collision frequency on the velocity and for $T_e \gg T$, the gradients of the temperatures of electrons and the gas are related by the following equation: $\nabla T = (1 + n)\nabla T_e$. Hence, the heat conductivity associated with electrons is

$$\varkappa = \frac{N_e}{3(1+n)} \left\langle \frac{mv^2}{2T_e} \frac{v^2}{\nu_{el}} \left(\frac{mv^2}{2T_e} - \frac{5}{2}\right) \right\rangle$$

$$= \frac{4T_e N_e}{3\sqrt{\pi}\, m(1+n)} \int_0^\infty z^{5/2} \left(z - \frac{5}{2}\right) \exp(-z)\, \nu_{el}^{-1}(z)\, dz$$

where $z = mv^2/2T_e$.

Since the frequency of collisions between electrons and atoms depends on the collision velocity as $\nu_{el} \sim v^n$, we have $\nu_{el}(v) = \nu_T z^{n/2}$, where $\nu_T \equiv \nu_{el}\sqrt{2T/m}$. Hence we obtain

$$\varkappa = \frac{4T_eN_e}{3\sqrt{\pi}\,m(1+n)\,\nu_T}\int\limits_0^\infty z^{\frac{5-n}{2}}\left(z - \frac{5}{2}\right)\exp(-z)\,dz$$

$$= \frac{4T_eN_e}{3\sqrt{\pi}\,m\nu_T}\,\frac{\left(1 - \frac{n}{2}\right)}{(1+n)}\,\Gamma\left(\frac{7-n}{2}\right)$$

In particular, if $\nu_{el} = \text{const}$

$$\varkappa = \frac{5N_eT_e}{2m\nu_{el}}$$

If $\sigma^* = \text{const}$ (that is, if $n = 1$)

$$\varkappa = \frac{2T_eN_e}{3\sqrt{\pi}\,m\nu_T}$$

Problem 2.28. Calculate the heat conductivity of a weakly ionized plasma contained by dielectric walls.

In the calculation of the heat conductivity of a weakly ionized plasma in Problem 2.27 we assumed that the electric field, that can appear in a plasma owing to the presence of the temperature gradient, was zero. This is the case for the plasma confined within a metal shell in which any current freely flows in the walls. In this problem the walls are insulators and no current can flow across the plasma. This leads to the appearance of an electric field in the plasma, which compensates the current due to the temperature gradient.

The general equations for the heat flux $\mathbf{q} = N_e\left\langle\frac{mv^2}{2}\mathbf{v}\right\rangle$ and the electric current $\mathbf{j} = eN_e\langle\mathbf{v}\rangle$ in a weakly ionized plasma with the electron temperature gradient ∇T_e and the electric field $\mathbf{E}$ can be written in the following form:

$$\mathbf{q} = -\alpha_{11}\nabla T_e + \alpha_{12}e\mathbf{E}$$

$$\mathbf{j} = -\alpha_{21}\nabla T_e + \alpha_{22}e\mathbf{E}$$

Using the distribution functions for electrons in weakly ionized plasmas in the presence of an electric field or temperature gradient found in the solutions of the above problems we can derive the factors α_{ik}:

$$\alpha_{11} = \frac{4N_e}{3\sqrt{\pi}} \frac{T_e}{m} \int_0^\infty \frac{1}{\nu_{el}} z^{5/2} \left(z - \frac{5}{2}\right) \exp(-z)\, dz$$

$$\alpha_{12} = \frac{4N_e}{3\sqrt{\pi}} \frac{T_e}{m} \int_0^\infty \frac{1}{\nu_{el}} z^{5/2} \exp(-z)\, dz$$

$$\alpha_{21} = \frac{4eN_e}{3\sqrt{\pi m}} \int_0^\infty \frac{1}{\nu_{el}} z^{3/2} \left(z - \frac{5}{2}\right) \exp(-z)\, dz$$

$$\alpha_{22} = \frac{4\pi N_e}{3\sqrt{\pi m}} \int_0^\infty \frac{1}{\nu_{el}} z^{3/2} \exp(-z)\, dz$$

Here $\nu_{el} = N_a v \sigma_{ea}^*$ is the frequency of elastic collisions between electrons and atoms, N_a is the number density of atoms, σ_{ea}^* is the diffusion cross section of the scattering of electrons by atoms, $z = mv^2/2T_e$. The factors with $i \neq k$ describe the thermoelectric phenomena.

The heat conductivity can be found from the above formulas relating the heat flux and the electric current to the electric field strength and the temperature gradient. In our case of an insulated plasma the current in the plasma is zero; therefore, the presence of a temperature gradient gives rise to an electric field whose strength is given by the equation $e\mathbf{E} = \frac{\alpha_{21}}{\alpha_{22}} \nabla T_e$. The heat flux is

$$\mathbf{q} = -\alpha_{11} \nabla T_e (1 - \alpha_{12}\alpha_{21}/\alpha_{11}\alpha_{22})$$

We see that the heat conductivity of an insulated plasma differs from that of a plasma in a metal container by a factor of $(1 - \alpha_{12}\alpha_{21}/\alpha_{11}\alpha_{22})$. Using the above formulas for α_{ik}, calculate this factor for the following two cases of practical interest: the frequency of collisions between electrons and atoms is independent of the electron velocity ($\nu_{el} = \text{const}$); the cross section of elastic collisions between electrons and atoms is independent of the electron velocity ($\nu_{el} \sim z^{1/2}$).

In the first case the factor $\alpha_{21} = 0$, that is, the temperature gradient does not give rise to electric current and the heat conductivity is the same for plasmas in metal and insulated containers. In the second case $\alpha_{12}\alpha_{21}/\alpha_{11}\alpha_{22} = -1$, that is, the heat conductivity of an insulated weakly ionized plasma is twice that of a plasma in a metal container. Finally, if the degree of ionization of a plasma is high enough and the collision frequency ν is determined by the charged plasma particles, then $\nu_{el} \sim z^{-3/2}$, $\alpha_{12}\alpha_{21}/\alpha_{11}\alpha_{22} = 3/5$ and the heat conductivity of an insulated plasma is smaller by a factor of 2.5 than that of a plasma in a metal container.

Problem 2.29. Determine the energy distribution function for electrons in the range of energies high enough for the excitation of gas atoms if de-excitation of atoms occurs only as a result of collisions with electrons. A uniform electric field is applied to the gas and the mean electron energy is much lower than the excitation energy of atoms.

As follows from the system of equations (2.24), the equation for the spherically symmetric part of the distribution function in this case is

$$\frac{1}{3}\left(\frac{eE}{m}\right)^2 \frac{d}{dv}\left(\frac{v^2}{\nu_{el}}\frac{df_0}{dv}\right) = I_c(f_0)$$

This collision integral I_c should take into account the collisions resulting in excitation and de-excitation of atoms. Since the frequency of inelastic collisions is much higher than the frequency of collisions between electrons or the parameter $\frac{m}{M}\nu_{el}$ (here m is the electron mass and M is the atomic mass), our equation in the zero approximation must satisfy the condition that the number of fast electrons losing the energy owing to the excitation of the gas atoms is equal to the number of slow electrons acquiring energy owing to the de-excitation of atoms. Hence, for the electrons whose velocity is in the range from v to $v + dv$ we can write

$$\nu_{exc}(v)f(v)\, v^2\, dv = \nu_d(v')f(v')\, v'^2\, dv'$$

where ν_{exc} and ν_d are the frequencies of excitation and de-excitation of atoms by an electron impact, v' is the velocity of slow electrons, which is increased to the range from v to $v + dv$ owing to de-excitation of atoms, $v'^2 = v^2 - 2\Delta E/m$, and ΔE is the excitation energy of gas atoms.

The excitation frequency of atoms $\nu_{exc}(v) = N_0 v \sigma_{exc}(v)$, where N_0 is the number density of atoms in the ground state, and the de-excitation frequency $\nu_d(v) = N_{exc} v \sigma_d(v)$, where N_{exc} is the number density of excited atoms.

Using the principle of detailed balancing, we can derive a relationship between the cross sections of excitation (σ_{exc}) and de-excitation (σ_d) of atoms by an electron impact [see Appendix 2, eq. (A2.9)]. Hence we can find the relationship between the distribution functions for slow and fast electrons:

$$\frac{N_0}{g_0} f_0(v) = \frac{N_{exc}}{g_{exc}} f_0\left(\sqrt{v^2 - \frac{2\Delta E}{m}}\right), \qquad v > \sqrt{\frac{2\Delta E}{m}}$$

Here g_0 and g_{exc} are the statistical weights of the respective atomic states. From the above we derive

$$f_0(v) = f_0\left(\sqrt{\frac{2\Delta E}{m}}\right)\frac{f_0\left(\sqrt{v^2-\frac{2\Delta E}{m}}\right)}{f_0(0)}$$

In particular, if the slow electrons have a Maxwellian distribution, the above formula is transformed into

$$f_0(v) = f_0\left(\sqrt{\frac{2\Delta E}{m}}\right)\exp\left(-\frac{mv^2}{2T_e}+\frac{\Delta E}{T_e}\right)$$

that is, the presence of excited atoms restores the Maxwellian distribution of electrons.

The above distribution of fast electrons is valid if the mean electron energy is lower than the excitation energy of the gas atoms, as is the case for not very high field strengths. Under such conditions the number density of fast electrons is low and the distribution function of slow electrons does not depend on the excitations of atoms. The conditions that the frequency of electron-electron collisions and the parameter $m\nu_{el}/M$ should be small compared with the frequency of inelastic collisions between fast ($v > \sqrt{2\Delta E/m}$) electrons and gas atoms are satisfied in a weakly ionized plasma. The results obtained in this problem show that the slope of the energy distribution function of electrons can be noticeably decreased beyond the excitation threshold.

Problem 2.30. Find the velocity distribution function for electrons in a gas to which a variable electric field is applied. The electron number density is small and the electron-electron collisions can be ignored.

As before, assume that the frequency ν_{el} of elastic collisions between electrons and the gas particles is higher than the frequency of inelastic collisions ν_{in}. As in the case of uniform external field, this means that the velocity distribution function of electrons is close to being spherically symmetric and can be expressed as

$$f(\mathbf{v}, t) = f_0(v, t) + v_x f_1(v, t)$$

where the x-axis is parallel to the field.

Let us assume that the electric field varies harmonically. Then the kinetic equation for the velocity distribution function of electrons is

$$\frac{\partial f}{\partial t} + \frac{eE}{m}\cos\omega t\frac{\partial f}{\partial v_x} = I_c(f)$$

where ω is the frequency of the field. Since ν_{el} is much higher than the frequency of inelastic collisions ν_{in}, we obtain the following system of equations for f_0 and f_1:

$$\frac{\partial f_0}{\partial t}+\frac{eE}{3m}\cos\omega t\,\frac{\partial v^3 f_1}{v^2\,\partial v}=I_c(f_0),\qquad \frac{\partial f_1}{\partial t}+\frac{eE}{mv}\cos\omega t\,\frac{\partial f_0}{\partial v}=-\nu_{el} f_1 \tag{2.30}$$

It is not easy to find a solution of this system for an arbitrary relationship between its parameters. We shall analyze two limiting cases. If the field frequency ω is small and $\omega\tau \ll 1$, where $1/\tau$ is the frequency of variation of the electron energy [$I_c(f_0) \sim f_0/\tau$], the terms $\partial f_0/\partial t$ and $\partial f_1/\partial t$ in eqs. (2.30) can be ignored in comparison with the terms $I_c(f_0)$ and $\nu_{el} f_1$, respectively. Under these conditions the equations for f_0 and f_1 have the same form as in the stationary case and the velocity distribution function is given by eqs. (2.24), where E is replaced by $E\cos\omega t$.

Another limiting case, when $\omega \gg 1$, is of practical interest. Since $1/\tau$ is the frequency of energy exchange between the electrons and the atoms, then the energy transferred from an electron to the atoms in one period is a small fraction of the electron energy. Therefore, the variation of the electron energy during one period of oscillation is due only to interaction with the field. Since only the electrons corresponding to the asymmetric part f_1 of the distribution function exchange energy with the field, the spherically symmetric part f_0 of the distribution function is independent of time.

These considerations can be used to solve the system of equations (2.30) for $\omega\tau \gg 1$. Expand the distribution function into a Fourier series:

$$f_0(v,t)=\sum_n f_{0,n}(v)\exp(i\omega nt),\qquad f_1(v,t)=\sum_n f_{1,n}\exp(i\omega nt)$$

Then eqs. (2.30) are transformed into

$$i\omega n f_{0,n}+\frac{eE}{6m}\,\frac{dv^3(f_{1,n+1}+f_{1,n-1})}{v^2\,dv}=I_c(f_{0,n})$$

$$i\omega n f_{1,n}+\frac{eE}{2m}\,\frac{d(f_{0,n-1}+f_{0,n+1})}{v\,dv}=-\nu_{el} f_{1,n}$$

According to the above arguments, in these equations we can retain the term $f_{0,0}$ for the spherically symmetric part of the distribution function and the terms $f_{1,1}$ and $f_{1,-1}$ for the asymmetric part. Other terms of the series can be expressed by the following order-of-magnitude relationships:

$$f_{0,\pm 2k}\sim\frac{f_{0,0}}{(\omega\tau)^k}\,\frac{(\omega^2+\nu_{el}^2)^{(k-1)/2}}{(2k)!!\prod_{m=1}^{k-1}[(2m+1)^2\omega^2+\nu_{el}^2]^{1/2}}$$

$$f_{0,\pm(2k+1)}\sim\frac{f_{1,\pm 1}}{(\omega\tau)^k}\,\frac{(\omega^2+\nu_{el}^2)^{k/2}}{(2k)!!\prod_{m=1}^{k}[(2m+1)^2\omega^2+\nu_{el}^2]^{1/2}}$$

Retaining the expansion terms $f_{0,0} \equiv f_0, f_{1,1}$ and $f_{1,-1}$, we can transform eqs. (2.30) into

$$\frac{eE}{6mv^2}\frac{dv^3(f_{1,1}+f_{1,-1})}{dv} = I_c(f_0)$$

$$(i\omega + \nu_{el})f_{1,1} + \frac{eE}{2mv}\frac{df_0}{dv} = 0 \qquad (2.31)$$

$$(-i\omega + \nu_{el})f_{1,1} + \frac{eE}{2mv}\frac{df_0}{dv} = 0$$

Hence we obtain the following equation for f_0:

$$\frac{1}{6}\left(\frac{eE}{m}\right)^2\frac{d}{v^2dv}\left[\frac{v^2\nu_{el}}{(\omega^2+\nu_{el}^2)}\frac{df_0}{dv}\right] = I_c(f_0)$$

In the special case, when $I_c(f_0)$ is determined by elastic collisions between electrons and the gas atoms, this equation yields

$$f_0(v) = C\exp\left\{-\int_0^v\left[T + \frac{e^2E^2M}{6m^2(\omega^2+\nu_{el}^2)}\right]^{-1} mv\,dv\right\} \qquad (2.32a)$$

where C is a normalization constant. The spherically asymmetric part of the distribution function in this special case is

$$f_1(v,t) = -\frac{eE}{m(\omega^2+\nu_{el}^2)v}(\nu_{el}\cos\omega t + \omega\sin\omega t)\frac{df_0}{dv} \qquad (2.32b)$$

Problem 2.31. Determine the electron temperature for a weakly ionized plasma to which an ultra high-frequency field is applied. The electron number density is high, so that the rate of the energy exchange between electrons is much higher than the rate of the energy exchange between electrons and atoms. The field frequency ω is high compared with the frequency of the energy exchange between an electron and the gas atoms.

To take into account the collisions between electrons, we should add the respective collision integral to the first of eqs. (2.31), while the other two equations remain unchanged. Since the rate of variation of the electron energy in collisions between electrons is higher than in collisions between electrons and atoms, the first of the equations of the system (2.31) shows that the electron-electron collision integral is zero, as was the case in Problem 2.22. This leads to the Maxwellian form for the energy distribution function for electrons (see Problem 2.22).

Since the second and third equations of system (2.31) remain unchanged, eq. (2.32b) is also applicable in this case and yields the drift velocity of electrons

$$w(t) = \frac{eE}{3T_e} \left\langle v^2 \frac{(\nu_{el} \cos \omega t + \omega \sin \omega t)}{\omega^2 + \nu_{el}^2} \right\rangle$$

Here T_e is the electron temperature and the angle brackets denote the averaging over velocities using the Maxwellian distribution function. Other symbols were defined in Problem 2.30.

Let us write the balance equation for the energy of an electron. Multiply the kinetic equation for the electron

$$\frac{\partial f}{\partial t} + \frac{eE}{m} \cos \omega t \frac{\partial f}{\partial v_x} = I_{ee} + I_{ea}$$

by the electron kinetic energy $mv^2/2$, integrate the result over electron velocities and average over time. According to eq. (1.15), we have $\int I_{ee} \frac{mv^2}{2} d\mathbf{v} = 0$, and then we obtain

$$\overline{\frac{\partial}{\partial t} \left\langle \frac{mv^2}{2} \right\rangle} + \overline{eEw \cos \omega t} = \omega t \int \frac{mv^2}{2} I_{ea} \, d\mathbf{v}$$

The bar above the symbols denotes averaging over time. The first term on the left-hand side of the equation is the mean variation of the mean electron energy per unit time and is, by definition, zero.

Using the above expression for the drift velocity of electrons and the relationships $\overline{\cos^2 \omega t} = 1/2$ and $\overline{\cos \omega t \sin \omega t} = 0$, we can find the second term which is the mean energy acquired by an electron from the field per unit time:

$$\overline{eEw \cos \omega t} = \frac{(eE)^2}{6T_e} \left\langle \frac{\nu_{el} \, v^2}{\omega^2 + \nu_{el}^2} \right\rangle$$

The left-hand side of the energy balance equation, which is the energy transferred from an electron the gas in elastic collisions between this electron and the gas atoms was calculated in Problem 2.22. Using this result, we obtain the temperature difference between electrons and the gas:

$$T_e - T = \frac{M}{6} \left(\frac{eE}{m} \right)^2 \frac{\left\langle \frac{\nu_{el} v^2}{\omega^2 + \nu_{el}^2} \right\rangle}{\langle \nu_{el} v^2 \rangle}$$

For low field frequencies ($\omega/\nu_{el} \ll 1$) this expression is identical to the result of Problem 2.22 if the field strength E of this problem is replaced with the effective field $E/\sqrt{2}$. For high field frequencies ($\omega/\nu_{el} \gg 1$) the difference between the electron and atom temperatures does not depend on the frequency of collisions between an electron and atoms:

$$T_e - T = \frac{M}{6}\left(\frac{eE}{m\omega}\right)^2$$

Problem 2.32. Derive the electron-electron collision integral (the so-called *Landau collision integral*).

This problem is considerably simplified by the fact that the impact parameters resulting in small-angle scattering make a major contribution to the diffusion cross section of collision of two electrons in an ideal plasma. Therefore, the variation of the electron momentum $\mathbf{p}$ in one collision is small compared with its value, and the electron-electron collision integral I_{ee} can be written as the right-hand side of the continuity equation:

$$I_{ee} = -\frac{\partial \mathbf{j}}{\partial \mathbf{p}}$$

However, in contrast to the Fokker-Planck equation (1.67), the quantity $\mathbf{j}$ here is a quadratic function of the velocity distribution of electrons.

Write down the electron-electron collision integral as

$$I_{ee} = -\int [f(\mathbf{p})f(\mathbf{p}') - f(\mathbf{p}+\Delta)f(\mathbf{p}'+\Delta')] \times W\left(\mathbf{p}+\frac{\Delta}{2}, \mathbf{p}'+\frac{\Delta'}{2}, \Delta, \Delta'\right) d\mathbf{p}'\, d\Delta \qquad (2.33)$$

Here Δ and Δ' are the variations of the momenta of the first and second electrons in collision, $W(\mathbf{p}+\Delta/2, \mathbf{p}'+\Delta'/2, \Delta, \Delta')$ is the probability per unit time per unit volume that electrons with the momenta $\mathbf{p}$ and $\mathbf{p}'$ collide and change their momenta to $\mathbf{p}+\Delta$ and $\mathbf{p}'+\Delta'$, so that $W d\Delta = |\mathbf{v}-\mathbf{v}'|\, d\sigma$, where $\mathbf{v}$ and $\mathbf{v}'$ are the velocities of the electrons and $d\sigma$ is the differential cross section of collision between them resulting in the variation of momentum by Δ. We shall write the probability W as a function of the half-sum and the difference of the momenta of a given electron before and after collision in order to make a better use of the symmetry of the probability W, which is the same for direct and reverse processes. We can write

$$\left[W\left(\mathbf{p}+\frac{\Delta}{2}, \mathbf{p}'+\frac{\Delta'}{2}, \Delta, \Delta'\right) = W\left(\mathbf{p}+\frac{\Delta}{2}, \mathbf{p}'+\frac{\Delta'}{2}, -\Delta, -\Delta'\right)\right]$$

It can be seen that the probability W is an even function of Δ. According to the momentum conservation law, we have $\Delta = -\Delta'$.

Expand the integrand in eq. (2.33) into a series, since the variation of the momentum in collision is small and $|\boldsymbol{\Delta}| \ll |\mathbf{p}|$. The zero-order terms in $\boldsymbol{\Delta}$ cancel out and the first-order terms yield

$$\int d\mathbf{p}'\, d\boldsymbol{\Delta} W \left(f' \frac{\partial f}{\partial p_i} \Delta_i + f \frac{\partial f'}{\partial p_i'} \Delta_i' \right)$$

Here $f \equiv f(\mathbf{p})$, $f' \equiv f'(\mathbf{p}')$, p_i is the respective vector component and the subscripts repeated twice imply summation. Since W is an even function in $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}'$, this integral over $d\boldsymbol{\Delta}$ is zero.

In the second-order approximation in $\boldsymbol{\Delta}$ we obtain

$$I_{ee} = \int d\mathbf{p}'\, d\boldsymbol{\Delta} \left[W \left(\frac{\Delta_i \Delta_k}{2} f' \frac{\partial^2 f}{\partial p_i \partial p_k} + \Delta_i \Delta_k' \frac{\partial f'}{\partial p_k'} \frac{\partial f}{\partial p_i} + \Delta_i' \Delta_k' f \frac{\partial^2 f'}{\partial p_i' \partial p_k'} \right) \right.$$
$$\left. + \frac{1}{2} \left(\Delta_i \frac{\partial W}{\partial p_i} + \Delta_i' \frac{\partial W}{\partial p_i'} \right) \left(\Delta_k \frac{\partial f}{\partial p_k} f' + \Delta_k' \frac{\partial f'}{\partial p_k'} f \right) \right]$$

Integrate the series of these terms by parts:

$$\frac{1}{2} \int d\mathbf{p}'\, d\boldsymbol{\Delta}\, \Delta_k \Delta_i' W \frac{\partial f'}{\partial p_i'} \frac{\partial f}{\partial p_k} + \frac{1}{2} \int d\mathbf{p}'\, d\boldsymbol{\Delta}\, \Delta_i' \Delta_k f' \frac{\partial W}{\partial p_i'} \frac{\partial f}{\partial p_k}$$
$$= \frac{1}{2} \int d\mathbf{p}' d\boldsymbol{\Delta}\, \Delta_k \Delta_i' \frac{\partial f}{\partial p_k} \frac{\partial}{\partial p_i'} (W f') = 0$$

$$\frac{1}{2} \int d\mathbf{p}'\, d\boldsymbol{\Delta}\, \Delta_i' \Delta_k' f W \frac{\partial^2 f'}{\partial p_i' \partial p_k'} + \frac{1}{2} \int d\mathbf{p}'\, d\boldsymbol{\Delta}\, \Delta_i' \Delta_k' f \frac{\partial W}{\partial p_i'} \frac{\partial f'}{\partial p_k'}$$
$$= \frac{1}{2} \int d\mathbf{p}'\, d\boldsymbol{\Delta}\, \Delta_i' \Delta_k' f \frac{\partial}{\partial p_i'} \left(W \frac{\partial f'}{\partial p_k'} \right) = 0$$

since $f(p') = df'/dp' = 0$ for $p' \to \infty$. This yields

$$I_{ee} = \frac{1}{2} \int d\mathbf{p}'\, d\boldsymbol{\Delta} \left[\Delta_i \Delta_k W f' \frac{\partial^2 f}{\partial p_i \partial p_k} + \Delta_i \Delta_k' W \frac{\partial f'}{\partial p_k'} \frac{\partial f}{\partial p_i} \right.$$
$$\left. + \Delta_i \Delta_k \frac{\partial W}{\partial p_i} \frac{\partial f}{\partial p_k} f' + \Delta_i \Delta_k' \frac{\partial W}{\partial p_i} \frac{\partial f'}{\partial p_k'} f \right]$$
$$= \frac{1}{2} \frac{\partial}{\partial p_i} \int d\mathbf{p}'\, d\boldsymbol{\Delta} \left[\Delta_i \Delta_k W \frac{\partial f}{\partial p_k} f' + \Delta_i \Delta_k' W f \frac{\partial f'}{\partial p_k'} \right]$$
$$= \frac{1}{2} \frac{\partial}{\partial p_i} \int d\mathbf{p}'\, d\boldsymbol{\Delta}\, W \Delta_i \Delta_k \left(f' \frac{\partial f}{\partial p_k} - f \frac{\partial f'}{\partial p_k'} \right)$$

Thus, the electron-electron collision integral is the divergence of a certain flux vector, as it follows from the continuity equation:

$$I_{ee} = -\frac{\partial j_i}{\partial p_i} \tag{2.34}$$

This flux is

$$j_i = \int d\mathbf{p}' \left(f \frac{\partial f'}{\partial p'_k} - f' \frac{\partial f}{\partial p_k} \right) \mathscr{D}_{ik} \tag{2.35}$$

where

$$\mathscr{D}_{ik} = \int \frac{\Delta_i \Delta_k}{2} W \, d\Delta$$

This result is of a general nature. It is valid if a collision between particles results in a small variation of their momenta. Let us find the tensor $\mathscr{D}_{ik}$ for electrons in a plasma. For large impact parameters ρ the variation of the momentum of an electron in collision with another electron is small and given by

$$\Delta_i = -\int_{-\infty}^{+\infty} \frac{dU}{dx_i} dt = \int_{-\infty}^{+\infty} \frac{e^2}{r^3} \rho_i \, dt = \frac{2e^2 \rho_i}{\rho^2 g}$$

where $U = e^2/r$ is the potential of interaction between two electrons, r is the distance between the electrons, x_i is the projection of the radius vector $\mathbf{r}$ on a given axis, ρ_i is the projection of $\boldsymbol{\rho}$ on this axis, and g is the relative velocity of collision. Hence we obtain

$$\mathscr{D}_{ik} = \int \frac{\Delta_i \Delta_k}{2} W \, d\Delta = \int \frac{\Delta_i \Delta_k}{2} g \, d\sigma = \frac{2e^4}{g} \int \frac{\rho_i \rho_k}{\rho^4} d\sigma$$

To find the tensor $\mathscr{D}_{ik}$ let us take the x-axis parallel to the relative velocity g. Since the vector $\boldsymbol{\rho}$ is perpendicular to g, we have $\rho_x = 0$ and $\mathscr{D}_{xx} = \mathscr{D}_{xy} = \mathscr{D}_{xz} = 0$. Furthermore, since $d\sigma = \rho \, d\rho \, d\varphi$ and $\rho_y = \rho \cos \varphi$, $\rho_z = \rho \sin \varphi$, we obtain

$$\mathscr{D}_{xz} \sim \int_0^{2\pi} \rho_y \rho_z \, d\varphi = \int_0^{2\pi} \frac{1}{2} \rho^2 \sin 2\varphi \, d\varphi = 0$$

The nonzero tensor components are

$$\mathscr{D}_{yy} = \mathscr{D}_{zz} = \frac{2\pi e^4}{g} \int_{\rho_1}^{\rho_2} \frac{d\rho}{\rho}$$

This integral is divergent at the upper and lower limits. The divergence at the lower limit is associated with the approximation used here, since the above equations are not valid for large scattering angles. The lower limit of integration should be the impact parameter of the collision for which the interaction potential e^2/ρ_1 is of the order of the kinetic energy ε of the colliding electrons, so that the scattering angle is of the order of unity:

$$\rho_1 \approx e^2/\varepsilon$$

The upper limit of integration should be the Debye-Hückel radius $r_D = \sqrt{T/8\pi Ne^2}$ of charge screening, so that the potential of interaction between electrons differs from the Coulomb potential. Hence we obtain

$$\mathscr{D}_{yy} = \mathscr{D}_{zz} = \frac{2\pi e^4}{g} \ln \Lambda, \tag{2.36a}$$

where $\ln \Lambda \approx \ln (\bar{\varepsilon} r_D/e^2) = \ln (T^3/Ne^6)^{1/2}$ is the Coulomb logarithm which is defined here to a factor of about unity in the parentheses *. Therefore, the above result is valid if $\ln \Lambda \gg 1$. This condition is satisfied in the ideal plasma.

Let us now use an arbitrary frame of reference. The tensor can be constructed on the relative velocity vector g since it is independent of other vectors after integration with respect to $d\sigma$. Since $\mathscr{D}_{ik}$ is a symmetric tensor, it can be constructed on combinations of the type $g_i g_k$ or $g^2 \delta_{ik}$ ($\delta_{ik} = 0$ for $i \neq k$ and $\delta_{ik} = 1$ for $i = k$). Using the above expression for $\mathscr{D}_{ik}$ in the case when one of the coordinate axes is parallel to g, we obtain in an arbitrary frame of reference

$$\mathscr{D}_{ik} = 2\pi e^4 \ln \Lambda \frac{g^2 \delta_{ik} - g_i g_k}{g^3} \tag{2.36b}$$

Problem 2.33. The velocity distribution of electrons has the Maxwellian form in the major range of velocities. Derive the electron-electron collision integral.

This can be done by using the Landau collision integral given by eqs. (2.34) and (2.35) by replacing in them $f'(\mathbf{v})$ with the Maxwellian distribution function. However, we shall use another approach. First, note that, in contrast to the Landau collision integral, the electron-electron collision integral in this case is a linear functional of the distribution function for a test electron, since its partner in collision has the Maxwellian distribution function. Furthermore, the variation of energy in one collision is small, since the scattering angles for electrons are mainly small. Therefore, the electron-electron collision integral in this case can be expressed as the right-hand side of the Fokker-Planck equation [see eq. (1.73)]:

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* In the ideal plasma which is dealt with here the Debye-Hückel radius r_D is considerably larger than the mean distance ($\sim N^{-1/3}$) between the charged particles. Therefore, the quantity $\mathscr{D}_{ik}$ is determined, to a considerable extent, by simultaneous collisions of several electrons. Since we make use of the model with binary collisions of electrons, we cannot hope for a better accuracy. For instance, if we calculate the tensor $\mathscr{D}_{ik}$ using the interaction potential $(e^2/r)\exp(-r/r_D)$ and finding the exact integral with respect to $d\rho$, the accuracy of the above result will not be improved.

$$I_c(f) = \frac{1}{\sqrt{\varepsilon}} \frac{\partial}{\partial \varepsilon} \left[\sqrt{\varepsilon}\, B(\varepsilon) \left(\frac{\partial f}{\partial \varepsilon} + \frac{f}{T_e} \right) \right]$$

where ε is the electron energy, and T_e is the electron temperature.

If the electron temperature is considerably higher than its thermal energy, we can use for $B(\varepsilon)$ the expression found in Problem 1.31 for electron-atom collisions under the assumption that the velocity of the test electron is much higher than the velocity of atoms. Since in this case the velocity of the test electron is much higher than the velocity of its partner, we can use eq. (1.74) here and write

$$I_c(f) = \frac{1}{v^2} \frac{\partial}{\partial v} \left[v^3 \nu \left(f + \frac{T_e}{mv} \frac{\partial f}{\partial v} \right) \right] \tag{2.37}$$

where $\nu = N_e v \sigma^*$ is the frequency of electron-electron collisions, and N_e is the electron number density.

Calculating the diffusion cross section σ^* of the electron-electron scattering we shall assume that electrons travel in classical trajectories and the ions are stationary. Then the small-angle scattering results in the following variation of the electron momentum in the direction perpendicular to it:

$$\Delta p_\perp = \int_{-\infty}^{+\infty} F_\perp \, dt = - \int_{-\infty}^{+\infty} \frac{\rho}{R} \frac{\partial U}{\partial R} dt$$

Here $\mathbf{F} = -\partial U / \partial \mathbf{R}$ is the force with which the ion acts on the electron, $U = e^2/R$ is the Coulomb potential of interaction between the electron and the ion, R is the distance between them, t is time, and ρ is the impact parameter. For small scattering angles the free motion equation $R^2 = \rho^2 + v^2 t^2$ is valid, where v is the relative velocity of collision which is equal here to the velocity of the test electron. Hence

$$\Delta p_\perp = \int_{-\infty}^{+\infty} \frac{\rho e^2}{R^3} dt = \int_{-\infty}^{+\infty} \frac{\rho e^2 \, dt}{(\rho^2 + v^2 t^2)^{3/2}} = \frac{2e^2}{\rho v}$$

and the scattering angle

$$\theta = \frac{\Delta p_\perp}{v} = \frac{2e^2}{\rho v^2} = \frac{2e^2}{\varepsilon \rho}$$

where $\varepsilon = mv^2/2$ is the electron energy, m is the electron mass and $\mu = m/2$ is the reduced mass of two electrons.

Since the differential cross section of scattering is $d\sigma = 2\pi\rho\,d\rho$, the expression for the diffusion cross section of scattering shows that σ^* is divergent for small impact parameters:

$$\sigma^* = \int (1 - \cos\theta)\, 2\pi\rho\, d\rho \approx \int \frac{\theta^2}{2} 2\pi\rho\, d\rho = \frac{4\pi e^4}{\varepsilon^2} \int\limits_{\rho_{\min}}^{\rho_{\max}} \frac{d\rho}{\rho}$$

For small impact parameters the lower limit of the integral should correspond to the scattering angle of the order of unity, since beyond it the above expansion is not valid. Thus, $e^2/\varepsilon\rho_{\min} \sim 1$ and $\rho_{\min} \sim e^2/\varepsilon$. The upper limit of the integral should be of the order of r_D (the Debye-Hückel radius for plasma), since at this distance the potential of interaction between the electron and the ion loses its Coulomb form owing to screening by charged plasma particles. Hence, we obtain

$$\sigma^* = \frac{4\pi e^4}{\varepsilon^2} \ln \Lambda \qquad (2.38)$$

where $\ln \Lambda = \ln \frac{e^2}{\varepsilon r_D} \gg 1$ is the Coulomb logarithm.

Using the equation (2.38) for the diffusion cross section of scattering of a fast electron by slow plasma electrons, we can transform eq. (2.37) for the electron-electron collision integral into

$$I_c(f) = \frac{16\pi e^4 N_e \ln \Lambda}{m^2 v^2} \frac{\partial}{\partial v}\left(f + \frac{T_e}{mv}\frac{\partial f}{\partial v}\right) \qquad (2.39)$$

This equation is valid for electron velocities much higher than the thermal velocity.

Problem 2.34. Identify the conditions making possible to separate determination of the electron and atom temperatures in a weakly ionized plasma.

The electron temperature can be reasonably defined in those cases when the velocity distribution function of electrons has the Maxwellian form. This is the case when the frequency of the electron-electron collisions is high enough. Indeed, the kinetic equation (1.2) for the electron distribution function is

$$\frac{e\mathbf{E}}{m} \frac{\partial f}{\partial \mathbf{v}} = I_{ee} + I_{ea}$$

where I_{ee} is the electron-electron collision integral, and I_{ea} is the electron-atom collision integral. In Problem 2.21 it was shown that for a high electron-electron collision frequency the kinetic equation in the zero approximation is

$$I_{ee} = 0$$

yielding the Maxwellian distribution for electrons but with a temperature differing from the gas temperature. This means that the electron temperature can be defined if $\nu_{ee} \gg \nu_{ea}$, where ν_{ee} is the electron-electron collision frequency, and ν_{ea} is the electron-atom collision frequency. In particular, for elastic collisions between electrons and atoms we obtain

$$\nu_{ee} \sim N_e \frac{e^4}{T_e^2} \ln \Lambda \sqrt{\frac{T_e}{m}}, \quad \nu_{ea} \sim N_a \frac{m}{M} \sqrt{\frac{T_e}{m}} \sigma_{el}$$

$$N_e \gg N_a \frac{m}{M} \frac{\sigma_{el} T_e^2}{e^4 \ln \Lambda}$$

where N_a and N_e are the number densities of atoms and electrons, respectively, T_e is the electron temperature, $\ln \Lambda$ is the Coulomb logarithm, σ_{el} is the cross section of elastic collisions between electrons and atoms, and M is the atomic mass. Even through in a weakly ionized plasma $N_e \ll N_a$, the range of parameters in which this relationship is valid is wide, since $T_e^2 \sigma_{el}/e^4 \Lambda \ll 1$ and $m/M \ll 1$.

Problem 2.35. The frequency of electron-electron collisions in a weakly ionized gas is considerably higher than the frequency of elastic collisions between electrons and atoms. The excitation energy ε_0 of an electron level of the atom is considerably higher than the thermal electron energy. The excitation cross section of this level is large enough, so that if the energy of an electron is higher than ε_0, it excites an atom. Find the frequency of excitation of atoms by an individual electron assuming that de-excitation does not involve electrons.

The sought excitation frequency is independent of the frequency of inelastic collisions between an electron and atoms. Indeed, the frequency of inelastic collisions is so high that the time interval between two successive inelastic collisions of a test electron with atoms is determined by the rate with which the electron accumulates energy owing to electron-electron collisions. On the other hand, owing to the high frequency of inelastic collisions, the velocity distribution function for electrons sharply drops with increasing electron energy beyond the energy ε_0. Hence, the frequency ν_{ea} of inelastic collisions with atoms can be found from the equation

$$\nu_{ea} N_e = -\int_{v_0}^{\infty} 4\pi v^2 \, dv \, \frac{\partial f}{\partial t}$$

where $v_0 = (2\varepsilon_0/m)^{1/2}$, and m is the electron mass. The kinetic equation yields the electron-electron collision integral: $I_c(f) = \partial f/\partial t$. Using eq. (2.39), we obtain from the above equation

$$\nu_{ea} N_e = -\int_{v_0}^{\infty} 4\pi v^2 \, dv \, I_c(f) = 64\pi^2 e^4 N_e m^{-2} \ln \Lambda \left(f + \frac{T_e}{mv} \frac{\partial f}{\partial v} \right) \Bigg|_{v_0}$$

Let us determine the velocity distribution function for electrons. As shown above, it is the solution of the equation

$$I_c(f) = 0$$

with the boundary condition $f(v_0) = 0$. In the main range of electron velocities the velocity distribution function has the Maxwellian form:

$$f(v) = N_e \left(\frac{m}{2\pi T_e}\right)^{3/2} \exp\left(-\frac{mv^2}{2T_e}\right)$$

The solution of the above equation with the given boundary conditions is

$$f(v) = N_e \left(\frac{m}{2\pi T_e}\right)^{3/2} \left[\exp\left(-\frac{mv^2}{2T_e}\right) - \exp\left(-\frac{mv_0^2}{2T_e}\right)\right]$$

This solution is valid for $mv_0^2/2T_e \gg 1$, which is the condition for the distribution function to have the Maxwellian form in the main range of velocities. Hence, the frequency of inelastic collisions between an electron and atoms is

$$\nu_{ea} = 16\sqrt{2\pi}\, T_e^{-3/2} m^{-1/2} e^4 N_e \ln \Lambda \exp\left(-\frac{\varepsilon_0}{T_e}\right)$$

Problem 2.36. Determine the energy distribution function for electrons in the energy range corresponding to the excitation of atoms, if the atomic excitation energy ε_0 is much higher than the thermal energy of electrons. The frequency of inelastic collisions between electrons and the gas atoms is much lower than the frequency of collisions between electrons. To simplify the problem, assume that the rate constant of excitation of atoms by an electron impact is independent of the electron energy. Find the frequency of atom excitation by an individual electron.

We have to solve the kinetic equation for the electron distribution function taking into account the elastic electron-electron collisions and inelastic electron-atom collisions. This equation can be written as

$$I_{ee}(f) - \nu_{exc} f = 0$$

where f is the energy distribution function for electrons, $I_{ee}(f)$ is the electron-electron collision integral, and ν_{exc} is the frequency of atom excitation by an electron impact. The second term in this equation should be taken into account for the electron energies exceeding the threshold energy. We assume here that de-excitation of atoms does not involve electrons and this assumption is not reflected in the kinetic equation.

Using the expression for the electron-electron collision integral derived in Problem 2.33 [eq. (2.37)], we can rewrite the above equation as

$$\frac{1}{v^2}\frac{d}{dv}\left[v^3 \nu \left(f + \frac{T_e}{mv}\frac{df}{dv}\right)\right] - \nu_{exc} f = 0$$

where ν is the electron-electron collision frequency and T_e is the electron temperature. Note that $v^3\nu$ is independent of the electron velocity. Express the electron distribution function as $f(v) = \varphi(v)g(v)$, where $\varphi(v)$ is the Maxwellian distribution function, so that $d\varphi/dv = -mv\,\varphi/T_e$. Then we obtain

$$\frac{d^2g}{d\varepsilon^2} - \frac{1}{T_e}\frac{dg}{d\varepsilon} - \frac{\nu_{exc}g}{\nu\varepsilon T_e} = 0$$

where $\varepsilon = mv^2/2$.

Let us rewrite this equation in terms of the dimensionless parameters $x = \varepsilon/T_e$ and $\alpha = \nu_{exc}T_e/\nu\varepsilon_0$. According to the conditions of this problem, $x_0 = \varepsilon_0/T_e \gg 1$. Let us solve the resulting equation in the energy range $\varepsilon \sim \varepsilon_0$. We obtain the following two equations:

$$\frac{d^2g}{dx^2} - \frac{dg}{dx} = 0, \qquad x \leqslant x_0$$

$$\frac{d^2g}{dx^2} - \frac{dg}{dx} - \alpha g = 0, \qquad x > x_0$$

Since for $x \sim 1$ the electron distribution function has the Maxwellian form, for $x \to 0$ we have $g = 1$.

The above equations yield

$$g = C_1 - C_2 \exp(x), \qquad x \leqslant x_0$$

$$g = C_3 \exp(a_1 x) + C_4 \exp(a_2 x), \qquad x > x_0, \quad a_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \alpha}$$

Physical considerations suggest that the electron distribution function is lower than the Maxwellian distribution (that is, $g \leqslant 1$). Moreover, $g(x)$ monotonically decreases with increasing x. These features of the distribution are due to the inelastic processes which decrease the number of electrons at the tail of the distribution function. Since $g(x)$ decreases with increasing x, we have $C_4 = 0$. Owing to this fact and the positive sign of $g(x)$, we have $C_1 \sim C_2 \exp(x_0)$, and since $x_0 \gg 1$, we obtain $C_2 \ll C_1$. Therefore, the condition $g(0) = 1$ yields $C_1 = 1$. Finally, the solution of the kinetic equation can be written as

$$g = \begin{cases} 1 - C_2 \exp(x), & x \leqslant x_0 \\ C_3 \exp\left[\left(\frac{1}{2} - \sqrt{\frac{1}{4} + \alpha}\right) x\right], & x > x_0 \end{cases}$$

Since the distribution function and its derivative are continuous, the backward and forward logarithmic derivatives of the distribution function at the point x_0 are identical, and we obtain

$$\frac{C_2 \exp(x_0)}{1 - C_2 \exp(x_0)} = \sqrt{\frac{1}{4} + \alpha} - 1$$

Hence we find

$$g(x) = \begin{cases} 1 - \dfrac{\alpha}{\sqrt{1/4 + \alpha} + 1/2 + \alpha} \exp(x - x_0), & x \leqslant x_0 \\ \dfrac{\sqrt{1/4 + \alpha} + 1/2}{\sqrt{1/4 + \alpha} + 1/2 + \alpha} \exp[-(\sqrt{1/4 + \alpha} - 1/2)(x - x_0)], & x > x_0. \end{cases}$$

A comparison with Problem 2.35 shows that the condition of intense excitation of atoms by an electron impact is

$$\alpha = \frac{\nu_{exc} T_e}{\nu(\varepsilon_0)\varepsilon_0} \gg 1$$

Using the above expressions for the distribution function, we can now find the mean frequency $\overline{\nu}_{ea}$ of atomic excitation by an individual electron:

$$N_e \overline{\nu}_{ea} = \int_{v_0}^{\infty} \nu_{exc} f(v) \cdot 4\pi v^2 \, dv = \int_{v_0}^{\infty} I_{ee}(f) \cdot 4\pi v^2 \, dv$$

$$= -4\pi v^3 \nu \left(f + \frac{T_e}{d\varepsilon} \frac{df}{} \right) \Bigg|_{\varepsilon_0} = -4\pi v^3 \nu \varphi \frac{dg}{dx} \Bigg|_{x_0}$$

$$= \frac{4\pi v_0^3 \nu \varphi(\varepsilon_0) \alpha}{\alpha + 1/2 + \sqrt{1/4 + \alpha}}$$

If α is large, $\overline{\nu}_{ea}$ is independent of the frequency of inelastic collisions between electrons and atoms and is identical to the result found in Problem 2.35. In the limit of small α the electron distribution function has the Maxwellian form for any electron energies. Therefore, the mean frequency of atomic excitation in collision with one electron is independent of the frequency of elastic collisions between electrons and can be found from transformed general expression:

$$N_e \overline{\nu}_{ea} = \int_{v_0}^{\infty} 4\pi v^2 \, dv \, \varphi(v) \, \nu_{exc} = 4\pi v_0 \frac{T_e}{m} \varphi(\varepsilon_0) \, \nu_{exc}$$

§ 2.4. PLASMA IN EXTERNAL FIELDS

Problem 2.37. Determine the contributions of ions and electrons to the conductivity of a weakly ionized gas.

When the external electric field is applied to a weakly ionized gas, the gas conductivity Σ is defined as the proportionality factor relating the electric current density $\mathbf{j}$ to the electric field strength $\mathbf{E}$ ($\mathbf{j} = \Sigma\mathbf{E}$). In the absence of a magnetic field the conductivity Σ of an isotropic gas is a scalar.

The electric current is the sum of two components — the electron current and the ion current:

$$\mathbf{j} = -eN_e\mathbf{w}_e + eN_i\mathbf{w}_i$$

Here $\mathbf{w}_e$ and $\mathbf{w}_i$ are the mean velocities of electrons and ions, and N_e and N_i are the number densities of electrons and ions which are identical for a quasineutral plasma.

Making use of the mobilities K_e and K_i of electrons and ions in a gas (defined by the relationship $\mathbf{w}_{e,i} = K_{e,i}\mathbf{E}$) and taking into account the fact that electrons and ions travel in opposite directions in an electric field, we obtain the following expression for the conductivity:

$$\Sigma = eN_e(K_e + K_i)$$

Let us use the Einstein relation which is exact for low electric fields and approximate for strong fields. We obtain $K \sim e\mathscr{D}/T$, where T is the mean kinetic energy of the particles. Hence, the contributions of electrons and ions to the plasma conductivity are proportional to their diffusion coefficients in a plasma. An order-of-magnitude estimate of the diffusion coefficient is

$$\mathscr{D} \sim v\lambda \sim v/N\sigma$$

where v is the characteristic velocity of a particle, $\lambda \sim 1/N\sigma$ is the mean free path of a particle in a gas, N is the gas density, and σ is the cross section of elastic collisions between the given species and the gas particles. The cross sections of elastic collisions of electrons and ions with the gas particles are of the same order of magnitude. Hence we can estimate the ratio between the diffusion coefficients $\mathscr{D}_e$ and $\mathscr{D}_i$ for electrons and ions:

$$\mathscr{D}_e/\mathscr{D}_i \sim v_e/v_i \sim \sqrt{M/m}$$

Here v_e and v_i are the characteristic velocities of electrons and ions, respectively, and m and M are the electron and ion masses.

We see that the contribution of ions to the conductivity of the gas is smaller by the factor of $\sqrt{M/m}$ than the contribution of electrons. Thus, the conductivity of a weakly ionized gas is determined practically by the motion of electrons.

Problem 2.38. Find the conductivity of a weakly ionized gas in crossed magnetic and variable electric fields, making use of the tau approximation (see Problem 1.19).

The tau approximation consists in replacing the collision integral with the quantity $-(f-f^0)/\tau$, where $1/\tau$ is the effective frequency of collisions between an electron and the gas atoms. The kinetic equation in this approximation is written as

$$\frac{\partial f}{\partial t}+\mathbf{v}\,\nabla f+\frac{\mathbf{F}}{m}\frac{\partial f}{\partial \mathbf{v}}=-\frac{f-f^0}{\tau}$$

where f^0 is the equilibrium distribution function in the absence of external fields. We shall assume the electron distribution function to be independent of the spatial coordinates. Multiplying the kinetic equation by $\mathbf{v}$ and integrating the result over the electron velocities, we obtain the following equation for the drift velocity $\mathbf{w}$ of an electron:

$$\frac{d\mathbf{w}}{dt}-\frac{\mathbf{F}}{m}=-\frac{\mathbf{w}}{\tau}$$

This equation describes the motion of an electron in a gas; the drag force $m\mathbf{w}/\tau$ is caused by collisions between the electron and the gas atoms.

In our case the external force is $\mathbf{F}=e\mathbf{E}\exp(i\omega t)+\frac{e}{c}[\mathbf{wH}]$. Writing the solution as $\mathbf{w}\exp(i\omega t)$, we obtain the following system of equations for the velocity components:

$$m\left(i\omega+\frac{1}{\tau}\right)w_x=eE+\frac{e}{c}w_yH$$

$$m\left(i\omega+\frac{1}{\tau}\right)w_y=-\frac{e}{c}w_xH$$

Here the x-axis is parallel to the electric field and the z-axis is parallel to the magnetic field. The solution of this system of equations is

$$w_x+iw_y=\frac{eE}{m}\left[\frac{1}{\tau}+i(\omega+\omega_H)\right]^{-1}$$

$$w_x-iw_y=\frac{eE}{m}\left[\frac{1}{\tau}+i(\omega-\omega_H)\right]^{-1}$$

Hence the gas conductivity is

$$\Sigma=\frac{j_x}{E}=\frac{N_e ew}{E}\left[\frac{1+i\omega\tau}{1+(\omega_H^2-\omega^2)\tau^2+2i\omega\tau}\right]\Sigma_0$$

where $\Sigma_0 = N_e e^2\tau/m$ is the conductivity in the uniform electric field in the absence of a magnetic field, N_e is the electron number density, and $\omega_H = eH/mc$ is the Larmor frequency. This equation shows that under steady-state conditions the application of a magnetic field results in a decrease in the gas conductivity according to the following relation:

$$\Sigma = \frac{\Sigma_0}{1 + \omega^2\tau^2}$$

This means that, if the collision frequency is low, an electron travels in circles and its drift velocity is lower.

Let $\omega_H\tau \gg 1$. If we vary the frequency of the electric field, we see that the conductivity has a peak at $\omega = \omega_H$: $\Sigma = \Sigma_0/2$. This effect is known as the *cyclotron resonance*. An electron rotates in a circle and the frequency of the electric field is equal to the rotation frequency. Therefore, the direction of momentum transferred from the electric field to the free electron always coincides with the direction of its motion, so that if there were no collisions, the electron energy would increase continuously (as the energy of the electron travelling in an electric field). Collisions with the gas atoms limit the electron energy and determine the gas conductivity.

Problem 2.39. An inert gas contains an admixture of alkali metal atoms with a small degree of ionization. Find the amount of admixture needed to provide for the maximum conductivity of the mixture at a given density of the inert gas, the gas temperature and the strength of the weak electric field if the system is in thermodynamic equilibrium.

The gas conductivity is

$$\Sigma = N_e eK$$

where N_e is the electron number density, and K is the electron mobility. If the degree of ionization of the admixture atoms is small, the Saha equation (2.14) shows that the electron number density N_e is of the order of $N_{ad}^{1/2}$, where N_{ad} is the number density of the admixture atoms. The electron mobility in weak fields derived from eq. (2.24 b) for the asymmetric part of the distribution function (2.20) is

$$K = \frac{8e}{3\sqrt{\pi m}} \int_0^\infty \frac{t^4 \exp(-t^2)\,dt}{\nu_{el}}, \quad t = \sqrt{\frac{mv^2}{2T}} \tag{2.40}$$

where $\nu_{el} = v(N_a\sigma_a^* + N_{ad}\sigma_{ad}^*)$ is the frequency of collisions between atoms and electrons, N_a is the number density of the inert gas atoms, and σ_a^* and σ_{ad}^* are the diffusion cross sections of collisions between electrons and atoms of the inert gas and the alkali metal, respectively. Thus, we obtain

$$\Sigma = AN_{\mathrm{ad}}^{\frac{1}{2}} \int_0^\infty \frac{t^3 \exp(-t^2)\, dt}{N_{\mathrm{ad}}\, \sigma_{\mathrm{ad}}^* + N_{\mathrm{a}}\sigma_{\mathrm{a}}^*}$$

where A is a constant. Differentiating this expression with respect to N_{ad}, we can find the number density of the admixture atoms providing for the maximum conductivity given by the following equation:

$$\int_0^\infty \frac{(N_{\mathrm{ad}}\sigma_{\mathrm{ad}}^* - N_{\mathrm{a}}\sigma_{\mathrm{a}}^*)}{(N_{\mathrm{ad}}\sigma_{\mathrm{ad}}^* + N_{\mathrm{a}}\sigma_{\mathrm{a}}^*)^2}\, t^3 \exp(-t^2)\, dt = 0$$

We obtain $N_{\mathrm{ad}} \sim N_{\mathrm{a}} \frac{\sigma_{\mathrm{a}}^*}{\sigma_{\mathrm{ad}}^*}$. Since the cross section σ_{a}^* of elastic collisions between electrons and the inert gas atoms is smaller than the cross section σ_{ad}^* of elastic collisions between electrons and the alkali metal atoms by approximately two orders of magnitude, the required content of the alkali metal admixture under optimum conditions is small.

Problem 2.40. Calculate the dielectric constant of a plasma in the tau approximation (see Problem 1.19).

Let us find the relationship between the dielectric constant and the conductivity of a plasma. The plasma conductivity Σ is defined as the proportionality factor between a component of the electric current density $\mathbf{j}$ and the electric field strength $\mathbf{E}$ in the Ohm law:

$$j_\alpha = \Sigma_{\alpha\beta}(\omega, k) \exp[i(\mathbf{kr} - \omega t)]\, E_\beta$$

The electric field strength (and the current density) here varies harmonically with coordinates and time (E_β is independent of coordinates and time); the subscripts α and β denote the x, y and z components of the respective vectors and the tensor.

As before, we shall assume that the electric properties of the plasma are determined by electrons, so that $\mathbf{j} = -eN_e\mathbf{w}$, where N_e is the electron number density, and $\mathbf{w}$ is the mean electron velocity. Hence we have

$$-eN_e w_a = \Sigma_{\alpha\beta}(\omega, k)\, E_\beta \exp[i(\mathbf{kr} - \omega t)]$$

The dielectric constant is defined by the following relation:

$$D_\alpha \equiv E_\alpha + 4\pi P_\alpha = \varepsilon_{\alpha\beta}(\omega, k)\, E_\beta \exp[i(\mathbf{kr} - \omega t)]$$

Here $\mathbf{D}$ is the electric induction vector, and $\mathbf{P}$ is the polarization of the unit volume of the plasma, that is, the dipole moment of the unit plasma volume, produced under the effect of the external electric field. The time dependence of the coordinate of an electron can be expressed as $\mathbf{r} = \mathbf{r}_0 + \mathbf{r}' \exp(-i\omega t)$, where $\mathbf{r}_0$ is independent of the external field and $\mathbf{r}'$ is determined by the motion of the electron under the effect of the field. Hence, the mean electron velocity in the electric field is

$$\mathbf{w} = \frac{d\mathbf{r}}{dt} = - i\omega\mathbf{r}' \exp(-i\omega t)$$

and the plasma polarization is

$$\mathbf{P} = -e \sum_k \mathbf{r}'_k \exp(-i\omega t) = - ieN_e\mathbf{w}/\omega$$

where the sum is taken over the electrons in the unit volume of the plasma. Then we obtain

$$-4\pi ieN_e w_\alpha/\omega = (\varepsilon_{\alpha\beta} - \delta_{\alpha\beta}) E_\beta \exp[i(\mathbf{kr} - \omega t)]$$

$$\delta_{\alpha\beta} = 1 \quad \text{for} \quad \alpha = \beta \quad \text{and} \quad \delta_{\alpha\beta} = 0 \quad \text{for} \quad \alpha \neq \beta$$

A comparison of the above equations yields the relationship between the dielectric constant tensor and the conductivity tensor in the plasma:

$$\varepsilon_{\alpha\beta} = \delta_{\alpha\beta} + \frac{4\pi i}{\omega} \Sigma_{\alpha\beta} \tag{2.41}$$

Assume that the conductivity of a weakly ionized gas is determined by electrons and use the conductivity tensor for a gas in a varying electric field and a uniform magnetic field parallel to the z-axis, which was derived in Problem 2.38. The resulting dielectric constant tensor is

$$\varepsilon_{\alpha\beta}(\omega, k = 0) - \delta_{\alpha\beta}$$

$$= \begin{vmatrix} \dfrac{\omega_2^0\tau(i - \omega\tau)}{[1+(\omega_H^2-\omega^2)\tau^2+2i\omega\tau]\omega} & \dfrac{i\omega_0^2\omega_H\tau^2}{[1+(\omega_H^2-\omega^2)\tau^2+2i\omega\tau]\omega} & 0 \\ \dfrac{i\omega_0^2\tau^2\omega_H}{[1+(\omega_H^2-\omega^2)\tau^2+2i\omega\tau]\omega} & \dfrac{\omega_0^2\tau(i - \omega\tau)}{[1+(\omega_H^2-\omega^2)\tau^2+2i\omega\tau]\omega} & 0 \\ 0 & 0 & \dfrac{i\omega_0^2\tau}{\omega(1 - i\omega\tau)} \end{vmatrix}$$

where $\omega_0 = (4\pi N_e e^2/m)^{1/2}$ is the plasma frequency, and $\omega_H = eH/mc$ is the Larmor frequency.

In the absence of a magnetic field the diagonal components of the dielectric constant tensor are

$$\varepsilon = 1 + \frac{\omega_0^2\tau(i - \omega\tau)}{\omega(1 + \omega^2\tau^2)}$$

and the other components are zero. In particular, for high electric field frequencies ($\omega\tau \gg 1$) we have $\varepsilon = 1 - \omega_0^2/\omega^2$. The dielectric constant is independent of the collision frequency, since the mean electron velocity is determined by the field frequency.

If the electric field frequency is low ($\omega\tau \ll 1$), the dielectric constant tensor is

$$\varepsilon_{\alpha\beta}(\omega, k=0) - \delta_{\alpha\beta} = \left|\begin{matrix} \dfrac{i\omega_0^2\tau}{\omega(1+\omega_H^2\tau^2)} & \dfrac{i\omega_0^2\omega_H\tau^2}{\omega(1+\omega_H^2\tau^2)} & 0 \\ \dfrac{i\omega_0^2\omega_H\tau^2}{\omega(1+\omega_H^2\tau^2)} & \dfrac{i\omega_0^2\tau}{\omega(1+\omega_H^2\tau^2)} & 0 \\ 0 & 0 & \dfrac{i\omega_0^2\tau}{\omega} \end{matrix}\right|$$

Problem 2.41. Find the relationship between the conductivity and the dielectric constant of a plasma using the tau approximation, if there are no external fields and the velocity distribution function of electrons is unperturbed.

In the tau approximation the velocity distribution function for electrons in the absence of external fields is the solution of the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\nabla f = -\frac{f - f_0}{\tau}$$

Assume that we applied to the plasma a weak electric field varying with coordinates and time as $\mathbf{E}\exp[i(\mathbf{kr} - \omega t)]$. The field gives rise to the electric current in the plasma. The proportionality factor relating the current density to the field strength is the plasma conductivity corresponding to the given field frequency and the wave vector.

Assume, as before, that the electric current in the plasma is carried by electrons. Write the velocity distribution function in the form

$$f = f^{(0)} + \delta f \exp[i(\mathbf{kr} - \omega t)]$$

The first term of this function is the solution of the above Boltzmann equation. In the first-order approximation the kinetic equation takes into consideration the electric field:

$$-i\omega\,\delta f + i\mathbf{kv}\,\delta f - \frac{e\mathbf{E}}{m}\frac{\partial f^{(0)}}{\partial \mathbf{v}} = -\frac{\partial f}{\tau}$$

This equation yields

$$\delta f = \frac{e\mathbf{E}}{m}\frac{\partial f^{(0)}}{\partial \mathbf{v}}\left(i\omega - i\mathbf{kv} - \frac{1}{\tau}\right)^{-1}$$

Thus, the weak electric field gives rise to the electric current with the density

$$\mathbf{j} = -eN_e\int \mathbf{v}f\,d\mathbf{v} = \frac{N_e e^2}{im}\int \frac{\mathbf{v}\left(\mathbf{E}\dfrac{\partial f^{(0)}}{\partial \mathbf{v}}\right)\exp\left[i(\mathbf{kr}-\omega t)\right]}{\omega - \mathbf{kv} + \dfrac{i}{\tau}}\,d\mathbf{v}$$

where N_e is the electron number density and the distribution function is normalized to unity $\left(\int f^{(0)}\,d\mathbf{v} = 1\right)$.

Finally, we obtain the conductivity tensor for the plasma:

$$\Sigma_{\alpha\beta}(\omega, k) = \frac{N_e e^2}{im}\int \frac{v_\alpha \dfrac{\partial f^{(0)}}{\partial v_\beta}\,d\mathbf{v}}{\omega - \mathbf{kv} + \dfrac{i}{\tau}}. \tag{2.42a}$$

Hence, according to eq. (2.41), the dielectric constant tensor is

$$\varepsilon_{\alpha\beta}(\omega, k) = \delta_{\alpha\beta} + \frac{\omega_0^2}{\omega}\int \frac{v_\alpha \dfrac{\partial f^{(0)}}{\partial v_\beta}\,d\mathbf{v}}{\omega - \mathbf{kv} + \dfrac{i}{\tau}} \tag{2.42b}$$

where $\omega_0 = (4\pi N_e e^2/m)^{\frac{1}{2}}$ is the plasma frequency.

In particular, for $\mathbf{k} = 0$, we can evaluate the integral in eq. (2.42b) by parts $\left(\int v_\alpha \dfrac{\partial f^{(0)}}{\partial v_\beta}\,d\mathbf{v} = -\delta_{\alpha\beta}\right)$ and obtain

$$\varepsilon_{\alpha\beta}(\omega, k=0) = \delta_{\alpha\beta}\left[1 - \frac{\omega_0^2}{\omega\left(\omega + \dfrac{i}{\tau}\right)}\right]$$

This expression is identical to the result derived above.

Problem 2.42. Calculate the rate of dissipation of the external field energy in a weakly ionized gas in crossed varying electric and uniform magnetic fields.

The energy absorbed in the unit volume of a weakly ionized gas per unit time is $\mathbf{jE}$, where $\mathbf{j}$ is the density of the electric current in the gas, generated by the external electric field $\mathbf{E}$. According to Ohm's law, in the uniform electric field $\mathbf{j} = \Sigma_0\mathbf{E}$, where Σ_0 is the conductivity of a weakly ionized gas in the uniform electric field. Hence we have

$$\mathbf{jE} = q_0 \equiv \Sigma_0 E^2$$

where the Joule heat q_0 is the energy absorbed in the unit volume of the gas per unit time.

The strength of the varying electric field can be expressed as

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{\sqrt{2}}\{\mathbf{E}\exp[i(\mathbf{kr} - \omega t)] + \mathbf{E}^*[\exp(-i(\mathbf{kr} - \omega t)]\}$$

This notation allows for the fact that $\mathbf{E}(\mathbf{r}, t)$ is a real quantity and the factor $1/\sqrt{2}$ is used to make $\mathbf{E}$ here to determine the mean density of electric energy in a plasma in the same way as is the case for the strength of the uniform electric field.

The Ohm's law in this case yields

$$j_\alpha(\mathbf{r}, t) = \frac{1}{\sqrt{2}}\{\Sigma_{\alpha\beta}(\omega, \mathbf{k})E_\beta \exp[i(\mathbf{kr}-\omega t)] + \Sigma_{\alpha\beta}(-\omega, -\mathbf{k})E^*_\beta \exp[-i(\mathbf{kr}-\omega t)]\}$$

Since the current density is real, we have

$$\Sigma_{\alpha\beta}(\omega, \mathbf{k}) = \Sigma^*_{\alpha\beta}(-\omega, -\mathbf{k})$$

The above equations show that the energy absorbed in unit volume per unit time is

$$\overline{jE} = \frac{1}{2}\Sigma_{\alpha\beta}(\omega, k)E^*_\alpha E_\beta + \frac{1}{2}\Sigma^*_{\alpha\beta}(\omega, k)E_\alpha E^*_\beta = \frac{1}{2}(\Sigma_{\alpha\beta} + \Sigma^*_{\beta\alpha})E^*_\alpha E_\beta\,.$$

The bar over the symbols denotes averaging over time and summation is implied over the repeated subscripts.

As we usually do, assume that the electric field is parallel to the x-axis. Then we have

$$\overline{\mathbf{jE}} = \frac{1}{2}(\Sigma_{xx} + \Sigma^*_{xx})\,|E|^2$$

Using the expression for the component of the conductivity tensor (see Problem 2.39), we obtain

$$\overline{\mathbf{jE}} = \frac{q_0}{2}\left[\frac{1 + i\omega\tau}{1 + (\omega_H^2 - \omega^2)\tau^2 + 2i\omega\tau} + \frac{1 - i\omega\tau}{1 + (\omega_H^2 - \omega^2)\tau^2 - 2i\omega\tau}\right]$$

where q_0 is the heat dissipated by the uniform electric field with the same energy density as our varying field and the other symbols are given in the previous problems.

Let us consider limiting cases. If there is no magnetic field ($\omega_H\tau \to 0$),

$$\overline{\mathbf{jE}} = \frac{q_0}{1 + \omega^2\tau^2}$$

In the uniform electric field we have ($\omega\tau \to 0$)

$$\overline{\mathbf{jE}} = \frac{q_0}{1+\omega_H^2\tau^2}$$

In both cases the dependences on the characteristic parameters can be seen to be similar.

In the case of cyclotron resonance ($\omega_H = \omega$) we have

$$\overline{\mathbf{jE}} = q_0 \frac{1+2\omega^2\tau^2}{1+4\omega^2\tau^2}$$

When $\omega\tau \gg 1$, the energy absorption at resonance is half that in the uniform field. When $\omega\tau \gg 1$, we obtain for a wide frequency range

$$\overline{\mathbf{jE}} = \frac{q_0}{2}\left[1+\frac{(\omega_H^2-\omega^2)^2\tau^2}{4\omega^2}\right]^{-1}$$

It can be seen that the width of the resonance peak is $\Delta\omega \sim 1/\tau \ll \omega$. In the vicinity of the resonance frequency $|\omega - \omega_H| \ll \omega$ this equation can be rewritten as

$$\overline{\mathbf{jE}} = \frac{q_0}{2[1+(\omega-\omega_H)^2\tau^2]}$$

Problem 2.43. Calculate the conductivity of a weakly ionized plasma if its degree of ionization is high enough for the conductivity to be determined by collisions between electrons and ions.

According to eq. (2.40), for weak fields the electron mobility in a gas is

$$K = \frac{8e}{3\sqrt{\pi}\, m}\int_0^\infty \frac{t^4 \exp(-t^2)\, dt}{\nu_{el}}$$

where $t = \sqrt{mv^2/2T}$, and ν_{el} is the frequency of elastic collisions between electrons and scattering particles. These particles are not electrons, since in electron-electron collisions the total momentum of electrons is conserved. Therefore, the frequency of collisions here should be the frequency of collisions between electrons and ions.

The frequency of collisions between electrons and ions, by definition, is

$$\nu = N_i v \sigma_{ei}^*$$

where N_i is the number density of ions, v is the electron velocity, and σ_{ei}^* is the diffusion cross section of electron scattering by ion. This cross section can be derived from the diffusion cross section (2.38) of collision of two electrons by changing in it the reduced mass of the colliding particles. We obtain

$$\sigma_{\mathrm{ei}}^{*} = \frac{\pi e^4}{\varepsilon^2} \ln \Lambda$$

where ε is the electron energy.

Substituting the expression for the diffusion cross section for scattering of an electron by an ion into the formula for the electron mobility and using the Maxwellian electron velocity distribution, we obtain

$$K = \frac{2(2T_{\mathrm{e}})^{3/2}}{N_{\mathrm{e}} e^3 m^{1/2} \pi^{3/2} \ln \Lambda}$$

where T_{e} is the electron temperature. Hence the conductivity of the quasineutral plasma is given by

$$\Sigma = N_{\mathrm{e}} eK = \frac{2\,(2T_{\mathrm{e}})^{3/2}}{e^2\, m^{1/2} \pi^{3/2} \ln \Lambda}$$

The above results are valid if the frequency of electron-ion collisions is considerably higher than the frequency of collisions between an electron and the neutral gas particles. Comparing the expressions for the collision frequencies we obtain the following condition:

$$\frac{N_{\mathrm{e}}}{N} \gg \frac{\sigma_{\mathrm{en}}^{*} T_{\mathrm{e}}^2}{\pi e^4 \ln \Lambda}$$

where N is the number density of the neutral gas particles, and σ_{en}^{*} is the characteristic diffusion cross section for scattering of an electron by the neutral gas particles. Owing to large Coulomb scattering cross section, scattering of electrons by ions is seen to become significant for a low degree of ionization of the gas ($N_{\mathrm{e}} \ll N$).

Problem 2.44. The translational velocity of electrons in a high-conductivity plasma is considerably higher than the translational velocity of ions. The electron current and the external conditions give rise to magnetic fields in the plasma, which affect its properties. Show that in the limit of high conductivity the magnetic field lines are "frozen" into the plasma, that is, they coincide with the electron current lines.

We shall describe the motion of the plasma by the continuity equation for electrons and the Euler, Poisson and Maxwell equations. The resulting system of equations describes the motion of plasma electrons and the variation of the fields in the plasma:

$$\frac{\partial N_{\mathrm{e}}}{\partial t} + \operatorname{div}(N_{\mathrm{e}} \mathbf{w}) = 0$$

$$\frac{\partial \mathbf{w}}{\partial t} + (\mathbf{w}\nabla)\,\mathbf{w} + \frac{\nabla p}{MN} - \frac{\mathbf{F}}{M} = 0$$

$$\operatorname{div} \mathbf{E} = 4\pi e(N_i - N_e)$$

$$\operatorname{curl} \mathbf{H} = -\frac{4\pi}{c}\, eN_e\mathbf{w} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$$

$$\operatorname{div} \mathbf{H} = 0$$

Here $\mathbf{w}$ is the translational velocity of electrons, and N_e and N_i are the number densities of electrons and ions, respectively. This system of equations is known as the system of equations of magnetohydrodynamics.

Under the conditions of this problem, when the ions can be assumed to be stationary, the electric field $\mathbf{E}' = \frac{1}{c}[\mathbf{wH}]$ is acting on electrons in the laboratory frame of reference. Since the plasma conductivity is high, this field gives rise to electric current in the plasma, which flows until the displacement of electrons produces the electric field

$$\mathbf{E} = -\frac{1}{c}[\mathbf{wH}]$$

which compensates the field of moving electrons. Substituting this field into the fifth equation of magnetohydrodynamics, we obtain

$$\frac{\partial \mathbf{H}}{\partial t} = \operatorname{curl}[\mathbf{wH}]$$

Compare the behaviour of the magnetic field described by this equation with the character of motion of electrons in a plasma. Using the last Maxwell equation div $\mathbf{H} = 0$, we can write

$$\begin{aligned}\operatorname{curl}\,[\mathbf{wH}] &= \mathbf{w}(\nabla\mathbf{H}) + (\mathbf{H}\nabla)\mathbf{w} - \mathbf{H}(\nabla\mathbf{w}) - (\mathbf{w}\nabla)\mathbf{H}\\ &= (\mathbf{H}\nabla)\mathbf{w} - \mathbf{H}\operatorname{div}\mathbf{w} - (\mathbf{w}\nabla)\mathbf{H}\end{aligned}$$

When we substitute into this equation div $\mathbf{w}$ derived from the continuity equation $\partial N_e/\partial t + N_e \operatorname{div} \mathbf{w} + \mathbf{w}\nabla N_e = 0$, we have,

$$\frac{\partial \mathbf{H}}{\partial t} = (\mathbf{H}\nabla)\,\mathbf{w} - (\mathbf{w}\nabla)\,\mathbf{H} + \frac{\mathbf{H}}{N_e}\frac{\partial N_e}{\partial t} + \frac{\mathbf{H}}{N_e}(\mathbf{w}\nabla N_e)$$

Dividing this equation by N_e and making transformations, we obtain

$$\frac{d}{dt}\left(\frac{\mathbf{H}}{N_e}\right)=\left(\frac{\mathbf{H}}{N_e}\nabla\right)\mathbf{w}$$

where the total derivative is taken at a point travelling with plasma and is equal to $d/dt = \partial/\partial t + (\mathbf{w}\nabla)$. This equation describes the character of plasma motion and the variation of the plasma field.

Consider an elementary plasma volume with the length $d\mathbf{l}$, initially parallel to the magnetic field, and the cross section area ds. The number of electrons in this volume is $N_e\, ds\, dl$ and the magnetic flux across it is $H\, ds$. We have to determine the variation of this flux with time.

Let the plasma velocity at one end of the segment $d\mathbf{l}$ be $\mathbf{w}$. Then at the other end of this segment the plasma velocity is $\mathbf{w}+(d\mathbf{l}\nabla)\mathbf{w}$, and the variation of the length of this segment during time δt is $(d\mathbf{l}\nabla)\mathbf{w}\,\delta t$. Hence, the length of the segment satisfies the equation

$$\frac{d}{dt}(d\mathbf{l}) = (d\mathbf{l}\nabla)\,\mathbf{w}$$

This equation is identical to the equation for $\mathbf{H}/N_e$. Since the vectors $d\mathbf{l}$ and $\mathbf{H}/N_e$ had the same direction at the starting moment, they remain parallel in the process of motion. Moreover, the ratio of these quantities remains constant in time: $H/(N_e\, dl) =$ const, that is, $H\, ds/(N_e\, dl\, ds) =$ = const. This suggests that the magnetic flux across an elementary volume of plasma containing a given number of electrons does not change with time. Thus, the magnetic field lines are "frozen" in plasma, that is, the electrons of a high-conductivity plasma travel along magnetic field lines.

Problem 2.45. A linear current $\mathscr{I}$ flows in a gas. Find the radius of the plasma column.

Assuming that the translational velocity w of electrons is much larger than that of ions, we can find the force acting on each plasma electron:

$$\mathbf{F} = -e\mathbf{E} = \frac{e}{c}\,[\mathbf{wH}]$$

Assume that the drift velocity w of electrons in the plasma is constant and parallel to the z-axis. Then the Euler equation is

$$\nabla p_e - \frac{1}{c}[\mathbf{jH}] = 0$$

where $\mathbf{j} = -eN_e\mathbf{w}$ is the electron current density, and p_e is the pressure in the plasma. Substituting the current density from the Maxwell equation

$\mathbf{j} = \frac{c}{4\pi}$ curl **H**, we obtain

$$\nabla p_e - \frac{1}{4\pi}[\operatorname{curl} \mathbf{H} \cdot \mathbf{H}] = 0$$

Since $[\operatorname{curl} \mathbf{H} \cdot \mathbf{H}] = (\mathbf{H}\nabla)\mathbf{H} - \frac{1}{2}\nabla \mathbf{H}^2$, we obtain

$$\nabla \left(p_e + \frac{H^2}{8\pi}\right) - \frac{1}{4\pi}(\mathbf{H}\nabla)\mathbf{H} = 0$$

The quantity $H^2/8\pi$ is known as the *magnetic pressure* and can be regarded as the pressure of the magnetic field. The above equation describes the spatial distribution of plasma and a magnetic field.

When a strong linear current flows in a gas, we shall assume that the current produces a plasma column in which the pressure is determined by the plasma electrons. This column has a cylindrical shape. Let us transform the above equation. Let $\mathbf{n}$ be the unit vector parallel to the magnetic field, that is, $\mathbf{H} = \mathbf{n}H$. We obtain

$$(\mathbf{H}\nabla)\mathbf{H} = H^2(\mathbf{n}\nabla)\mathbf{n} + \mathbf{n}H(\mathbf{n}\nabla)H = \frac{H^2}{R}\mathbf{n} + \nabla_{\parallel}\frac{H^2}{2}$$

where R is the curvature radius of the magnetic field line, which is infinite for the linear current, and the operator $\nabla_{\parallel}$ corresponds to the differentiation in the direction of the magnetic field. Hence we obtain

$$\nabla_{\perp}\left(p_e + \frac{H^2}{8\pi}\right) = 0$$

where $\nabla_{\perp}$ denotes the differentiation in the direction perpendicular to the magnetic field. This means that the total pressure, which is the sum of the plasma pressure p_e and the magnetic field pressure, is constant in space:

$$p_e + H^2/8\pi = \text{const}$$

Let us write down this condition for the boundary of a cylindrical plasma column. Assume that the magnetic field does not penetrate the plasma. This is the case if the current is concentrated mainly at the boundary of the plasma column. Then the pressure at the boundary inside plasma is p_e and outside plasma the total pressure is $H^2/8\pi$.

The magnetic field produced by the current $\mathcal{J}$ at the boundary of the plasma column is $H = 2\mathcal{J}/cr_0$, where r_0 is the column radius. Equating the pressure inside and outside the plasma boundary, we obtain

$$p_e = \frac{\mathcal{J}^2}{2\pi c^2 r_0^2}$$

and hence

$$r_0 = \frac{\mathcal{J}}{c}\sqrt{\frac{1}{2\pi p_e}}$$

Problem 2.46. A cylindrical tube with a radial electric field **E** and a longitudinal magnetic field **H** is used to generate a weakly ionized plasma in a gas containing two isotopes with the atomic masses m_1 and m_2. Calculate the difference between the radial currents of the ions of different isotopes in the limit of the strong electric and magnetic fields.

The motion equation for the ions of the first isotope is

$$m_1 \frac{d\mathbf{w}_1}{dt} = e\mathbf{E} + \frac{e}{c}[\mathbf{w}_1\mathbf{H}] - \mu_1 \mathbf{w}_1 \nu + \frac{m_1 w_{1\tau}^2 \boldsymbol{\rho}}{\rho^2}$$

where $\mathbf{w}_1$ is the mean velocity of the ions, μ_1 is the reduced mass of the ion and the gas particle, the first two terms on the right-hand side are the electric and magnetic forces acting on the ion, respectively, and the third term is the drag force due to collisions between the ions and the gas particles. If the reduced masses for two isotopes are close ($\mu_1 \approx \mu_2 = \mu$), the collision frequencies ν for the ions and gas particles are also similar, and we shall assume them to be identical. The last term in the above equation is the centrifugal force, so that $w_{1\tau}$ is the tangential velocity component, and $\boldsymbol{\rho}$ is the radius vector perpendicular to the cylinder axis.

The equation of motion for the ions of the second isotope has the same form as the equation for the ions of the first isotope, so that we shall consider only the latter equation omitting the subscripts denoting the number of the isotope and use the results for both ion species.

We shall solve the equation in the limit of strong fields, when the Larmor frequency $\omega_H = e\mathrm{H}/mc$ for the ions is considerably higher than the frequency of collisions between the ions and the gas particles

$$\omega_H \gg \nu$$

and the Larmor radius for the ion is much smaller than the characteristic dimensions of the system

$$w/\omega_H \ll \rho$$

The last condition makes it possible to ignore the dependence of the problem parameters on the distance ρ from the axis and the small forces acting on the ion and independent of its mass (for instance, the force due to nonuniform distribution of the ion density along the radius).

Using the above relationships, expand the ion velocity into a power series in $1/H$: $w = w^{(0)} + w^{(1)} + \ldots$. In the zero approximation the equation is transformed into

$$eE + \frac{e}{c} w_\tau^{(0)} H = 0$$

and its solution is $w_\tau^{(0)} = -c\frac{E}{H}$. In the first approximation the respective equations for the velocity components are

$$\left.\begin{aligned} \frac{e}{c} w_\tau^{(1)} H - \mu\nu w_\rho^{(1)} + \frac{m_1 (w_\tau^{(0)})^2}{\rho} = 0 \\ -\frac{e}{c} w_\rho^{(1)} - \mu\nu w_\tau^{(1)} = 0 \end{aligned}\right\}$$

where $w_\rho^{(1)}$ is the velocity component in the direction of the vector ρ. The solution of this system of equations with the condition $eH/\mu c \gg \nu$ is

$$w_\rho^{(1)} = \frac{\mu\nu c^2}{e^2 H^2} \frac{m_1 (w_\tau^{(0)})^2}{\rho} = m_1 \frac{\mu\nu c^4 E^2}{e^2 H^4 \rho}$$

A similar expression can be obtained for the mean velocity of the ions of the second isotope in the direction of the radius of the tube:

$$w_\rho^{(2)} = m_2 \frac{\mu\nu c^4 E^2}{e^2 H^4 \rho}$$

The process results in some separation of isotopes as the ions of the lighter isotope are nearer to the tube axis than the ions of the heavier isotope.

CHAPTER 3

WAVES IN PLASMA

The long-range Coulomb interaction between charged particles in plasma affects not only its static parameters but also the wave properties. The types of oscillatory plasma motion determined by the external fields and the spatial plasma distribution and the velocity distributions of charged particles can be highly varied. The role of oscillations in plasma is more significant than the role of acoustic vibrations in the processes occurring in gases owing to the long-range interaction of charged particles in plasma. Therefore, many processes involved in absorption and transformation of external energy and approaching equilibrium in plasma proceed via excitation of waves in it. The wide-spectrum vibrations in plasma can have considerable energy and affect the processes of approaching equilibrium as well as the averaged characteristics of plasma. Thus, waves in plasma play a particularly significant role.

§ 3.1. SMALL-AMPLITUDE OSCILLATIONS IN PLASMA

The small-amplitude oscillations in a plasma can be regarded as waves that do not affect the equilibrium properties of the plasma. Then the waves in the plasma can be considered as perturbations of its equilibrium properties and the presence of waves can be ignored in determining the equilibrium plasma characteristics. Similarly, we can ignore the interaction between waves of different wavelengths. Thus, each plasma parameter in the presence of waves can be expressed as

$$A = A_0 + A_1 \exp [i(\mathbf{kr} - \omega t)]$$

where A_0 is the parameter in the absence of waves, A_1 is the oscillation amplitude, $\mathbf{k}$ is the wave vector of the wave related to the wavelength by the equation $k = 2\pi/\lambda$, and ω is the wave frequency. In this approximation we have to find the relationship between the wave frequency ω and the wave vector $\mathbf{k}$, which is known as the *dispersion relation*. The theory ignoring the effect of oscillations on the properties of plasma and the interaction between waves is known as the *linear theory of oscillations.*

Problem 3.1. Derive the dispersion relation for the sound travelling in a stationary gas.

Let us make use of the continuity equation (1.16) and the Euler equation (1.18):

$$\frac{\partial N}{\partial t}+\frac{\partial}{\partial x_k}Nv_{0k}=0,\quad \frac{\partial v_{0k}}{\partial t}+\frac{1}{MN}\frac{\partial P_{jk}}{\partial x_j}=0 \tag{3.1}$$

Here $\mathbf{v}_0$ is the velocity of translational motion of the gas, N is the gas density, M is the mass of a gas particle, and P_{jk} is the pressure tensor which is given by eq. (1.17) for the stationary gas at thermodynamic equilibrium as $P_{jk} = MN\langle v_j v_k\rangle = p\delta_{jk}$, where p is the gas pressure.

Since the oscillation amplitude is small, we can write the parameters of the stationary gas in the form

$$N = N_0 + N'\exp[i(\mathbf{kr}-\omega t)],\quad \mathbf{v}_0 = \mathbf{v}'\exp[i(\mathbf{kr}-\omega t)],\quad p = p_0 +$$
$$+p'\exp[i(\mathbf{kr}-\omega t)]$$

where N_0 and p_0 are the density and the pressure of the gas in the absence of vibrations, and N', p' and $\mathbf{v}'$ describe the deviation from the equilibrium. Substituting these expressions into eqs. (3.1), we obtain the following system of equations:

$$-i\omega N' + ikN_0v' = 0,\quad -i\omega v' + \frac{1}{MN_0}ikp' = 0$$

Hence, the dispersion relation is

$$\omega^2 = \frac{k^2}{M}\frac{p'}{N'}$$

The relationship between p' and N' can be derived from the equation $pN^{-\gamma} = \text{const}$ describing the adiabatic variation of the gas parameters in the process of sound propagation (here $\gamma = c_p/c_V$, where c_p and c_V are the gas specific heats at constant pressure and constant volume, respectively). Hence we obtain $p'/p_0 = \gamma N'/N_0$. Making use of the equation of state $p_0 = N_0T$ for the ideal gas (T is the gas temperature), we can transform the dispersion relation into

$$\omega = \sqrt{\gamma Tk/M} \tag{3.2}$$

The sound velocity $\sqrt{\gamma T/M}$ in the gas is seen to be of the order of the thermal velocity of the gas particles.

Problem 3.2. Find the frequency of oscillations of the electronic component of the quasineutral plasma for infinite wavelength.

Apart from the wave types common to various mediums, specific types of oscillations can develop in plasma. We shall consider here one such type of oscillations, which is associated with the motion of electrons.

Physically, these vibrations can be described as follows. If all the electrons in a certain plasma region are displaced, this gives rise to electric fields which tend to return the electrons to the equilibrium distribution. Oscillations of the electrons under the effect of these fields result in the development of the plasma oscillations. In contrast to electromagnetic waves, the plasma oscillations are, like sound, longitudinal waves, since the direction in which the electric field strength oscillates is the same as the direction of wave propagation.

The above picture can be described by shifting all the electrons of infinite volume of plasma starting from the plane $x = 0$ to the left by the distance Δx. Then a positively charged plasma layer with the thickness Δx is formed near the plane $x = 0$ and a negatively charged plasma layer of the same thickness is formed at infinity. These charged layers give rise to an electric field which tends to displace the charged particles and to eliminate the charged layers, thus generating plasma oscillations. We shall consider the type of oscillations determined by the plasma electrons (the ion motion lags behind the electron motion and the ions are not involved in oscillations of this type).

To find the oscillation frequency, we shall make use of the reference frame in which the plasma to the left of the boundary $(x < 0)$ is at rest. The equation of motion for the electrons to the right of the boundary is

$$m\frac{d^2}{dt^2}(x + \Delta x) \equiv m\Delta\ddot{x} = -eE$$

where x is the coordinate of an electron, e and m are its charge and mass, and E is the electric field strength produced by the charged layers. We have to add the Poisson equation

$$\operatorname{div}\mathbf{E} \equiv \frac{dE}{dx} = \begin{cases} 4\pi N_0 e & \text{for } \ 0 \leqslant x \leqslant \Delta x \\ 0 & \text{for } \ \Delta x < x \end{cases}$$

where N_0 is the equilibrium number density of the charged particles in the plasma. Since in our reference frame there is no translational motion of plasma particles for $x < 0$, we have $E = 0$ for $x < 0$. Hence, the Poisson equation yields $E = 4\pi N_0 e \Delta x$ for $x > \Delta x$ and the equation of motion for electrons to the right of the boundary is

$$\Delta\ddot{x} + \frac{4\pi N_0 e^2}{m}\Delta x = 0$$

This equation describes oscillation of the electron component of plasma with respect to the boundary with the frequency

$$\omega_0 = \sqrt{\frac{4\pi N_0 e^2}{m}} \tag{3.3}$$

Oscillations of this type are known as the *plasma*, or *Langmuir*, *oscillations.*

Problem 3.3. Derive the dispersion relation for the plasma waves determined by the electron motion in a plasma. Ignore the collisions between electrons and the gas particles.

Write down the distribution function for plasma electrons as the sum of two terms $[f = f_0(\mathbf{v}) + f'(\mathbf{v})]$ the first of which is the distribution function in the absence of plasma oscillations and the second is due to plasma oscillations and is small in comparison with the first one. For small-amplitude oscillations the kinetic equation (1.5) yields the following equation for the distribution function f':

$$\frac{\partial f'}{\partial t} + v\nabla f' - \frac{eE'}{m}\frac{\partial f_0}{\partial \mathbf{v}} = 0 \tag{3.4}$$

The Poisson equation

$$\operatorname{div} \mathbf{E}' = -4\pi eN \int f' \, d\mathbf{v} \tag{3.5}$$

should be added to eq. (3.4). Here the distribution function is normalized to unity: $\int f \, d\mathbf{v} = 1$. In deriving eqs. (3.4) and (3.5) we assumed the absence of external fields. Let us express the dependence of f' and $\mathbf{E}'$ on time and coordinate in the form of the plane wave $\exp[i(\mathbf{kr} - \omega t)]$. The wave is longitudinal, that is, the vectors $\mathbf{E}'$ and $\mathbf{k}$ have the same direction. Eliminating f' and $\mathbf{E}'$ from the system of equations (3.4) and (3.5), we obtain the following dispersion relation for the plasma waves:

$$\frac{\omega_0^2}{k^2} \int \frac{\frac{\partial f_0}{\partial v_x} d\mathbf{v}}{v_x - \frac{\omega}{k}} = 1 \tag{3.6}$$

We shall assume that in the absence of external field the variables in the kinetic equation for the velocity distribution of electrons are separable and the distribution can be written down as $f(\mathbf{v}) = f(v_x)\,\varphi(v_\rho)$. Then the dispersion relation (3.6) can be transformed into

$$\frac{\omega_0^2}{k^2} \int_{-\infty}^{+\infty} \frac{\frac{\partial f_0}{\partial v_x} dv_x}{v_x - \frac{\omega}{k}} = 1$$

The integrand has a pole at $v_x = \omega/k$. The imaginary part of the integral can be found by taking the integration path circumventing the pole:

$$\frac{\omega_0^2}{k^2}\int_{-\infty}^{+\infty}\frac{\frac{\partial f_0}{\partial v_x}\,dv_x}{v_x-\frac{\omega}{k}}+i\pi\frac{\omega_0}{k^2}\frac{\partial f_0}{dv_x}\bigg|_{v_x=\frac{\omega}{k}}=1$$

Here the principal value of the first integral is taken. Expanding the integrand into a power series in small wave vector, we obtain

$$-\int_{-\infty}^{+\infty}\frac{\omega_0^2}{kv_x}\frac{\partial f_0}{\partial v_x}\,dv_x\sum_{n=0}^{\infty}\left(\frac{v_xk}{\omega}\right)^n+i\pi\frac{\omega_0^2}{k^2}\frac{\partial f_0}{\partial v_x}\bigg|_{v_x=\frac{\omega}{k}}=1$$

Taking the first terms of the series we have

$$\frac{\omega_0^2}{\omega^2}\left(1+3\frac{k^2}{\omega^2}\langle v_x^2\rangle\right)+i\pi\frac{\omega_0^2}{k^2}\frac{\partial f_0}{\partial v_x}\bigg|_{v_x=\frac{\omega}{k}}=1$$

Since this treatment is applicable for $k^2\langle v_0^2\rangle \ll \omega^2$, we obtain

$$\omega^2=\omega_0^2+3k^2\langle v_x^2\rangle+i\pi\frac{\omega_0^4}{k^2}\frac{\partial f_0}{\partial v_x}\bigg|_{v_x=\frac{\omega}{k}} \tag{3.7}$$

Problem 3.4. Find the condition of adiabaticity for a sound wave in a gas.

Propagation of the sound wave in a gas is adiabatic if there is no energy exchange between the regions of compressed and rarified gas which should have different temperatures. Estimate the time τ_{equil} needed for equalization of these temperatures and compare it with the characteristic time between compression and rarefaction of the gas at a given point. The latter time is of the order of $1/\omega$, where ω is the wave frequency. It can be seen that the condition of this problem corresponds to the condition $\omega\tau_{\text{equil}} \gg 1$.

The heat transfer equation is

$$\frac{\partial}{\partial t}(c_pT)=\nabla(\varkappa\nabla T)$$

where T is the gas temperature, $\varkappa$ is the heat conductivity, and c_p is the specific heat at constant pressure. The equation shows that the characteristic time in which heat is transferred at a distance r is of the order of $r^2c_p/\varkappa$. The specific heat is of the order of the number density of gas particles and the characteristic distance of temperature nonuniformity is of the order of the wavelength $1/k$ of the sound wave. Thus, the adiabaticity condition for the sound wave is

$$\tau_{\text{equil}} \sim N/k^2\varkappa, \quad \omega\tau_{\text{equil}} \sim N\omega/k^2\varkappa \gg 1$$

The heat conductivity $\varkappa$ of the gas is of the order of v_T/σ (see Problem 1.19), where σ is the characteristic cross section of collisions between the gas particles, $v_T \sim \sqrt{T/M}$ is the thermal velocity of the gas particles, and M is the mass of the gas particles. Defining the mean free path $\lambda \sim 1/N\sigma$ of the gas particles and using the dispersion relation between the frequency of the sound wave and its wave vector $\omega \sim v_T k$, we can rewrite the adiabaticity condition for the sound wave in the following form:

$$\lambda k \ll 1$$

Thus, propagation of the sound wave is adiabatic if the wavelength of the sound wave is much larger than the mean free path of the gas particles.

Problem 3.5. Find the adiabaticity condition for the plasma waves in a weakly ionized gas. The electron temperature is assumed to be different from the gas temperature and the rate of energy exchange between electrons is higher than the rate of energy exchange between electrons and atoms.

The adiabaticity condition for waves was derived in Problem 3.4 as

$$\varkappa k^2/c_p\omega \ll 1$$

We shall estimate the specific heat of the electron gas as $c_p \sim N_e$ and the heat conductivity due to electrons as $\varkappa \sim N_e v_e/N\sigma_{ea} \sim N_e v_e \lambda$, where N_e is the number density of electrons, v_e is their characteristic velocity, N is the number density of the gas particles, $\lambda \sim 1/N\sigma_{ea}$ is the mean free path of electrons in the gas, and σ_{ea} is the characteristic cross section of electron-atom collisions. These estimates yield

$$v_e\lambda k^2/\omega \ll 1$$

Let us define the Debye-Hückel radius $r_D \sim v_e/\omega_0$, where ω_0 is the plasma frequency, which is of the order of the frequency ω of plasma oscillations under consideration. Then we obtain the following adiabaticity condition for plasma oscillations:

$$\lambda r_D k^2 \ll 1$$

Note that plasma oscillations propagate in a gas if their frequency ω_0 is considerably higher than the frequency of elastic collisions between electrons and gas particles ($\omega_0 \gg N v_e \sigma_{ea} \sim v_e/\lambda$). This condition can be written as

$$\lambda \gg r_{\mathrm{D}}$$

Hence, at any rate, even if the wavelength of plasma oscillations is higher or comparable with the mean free path of electrons in a gas ($\lambda k \lesssim 1$), the adiabaticity condition for the plasma oscillations in a weakly ionized gas is satisfied.

Problem 3.6. Derive the dispersion relation for the other type of plasma oscillations in the absence of external fields, namely, the ion-acoustic wave (the ion sound).

The ion-acoustic waves involve motion of the heavier plasma species, that is, ions. Displacement of ions gives rise to an electric field which produces ion oscillations. The velocity of the lighter species, electrons, is much higher than that of the wave, so that their spatial distribution is determined by the Boltzmann distribution

$$N_{\mathrm{e}} = N_0 \exp(e\varphi/T_{\mathrm{e}}) = N_0\left(1 + \frac{e\varphi}{T_{\mathrm{e}}}\right)$$

where N_{e} is the number density of electrons, T_{e} is the electron temperature, N_0 is the mean number density of charged particles in the plasma, and φ is the potential of the electric field due to plasma oscillations. Since the amplitude of oscillations is small, we retained only the first term of the expansion of the distribution function.

Then we can write for the small-amplitude ion-acoustic wave

$$N_{\mathrm{i}} = N_0 + N_{\mathrm{i}}' \exp[i(kx - \omega t)], \quad w_{\mathrm{i}} = w_{\mathrm{i}}' \exp[i(kx - \omega t)],$$

$$\varphi = \varphi' \exp[i(kx - i\omega t)]$$

where N_{i} is the number density of ions, w_{i} is the mean velocity of ions, and φ is the electric field potential. The quantities $N_{\mathrm{i}}', w_{\mathrm{i}}'$ and φ' are the oscillation amplitudes of the respective parameters. Using the relationships between the plasma ion parameters, we can find the dispersion relation between the frequency ω and the wave vector k of the ion-acoustic wave.

Substitute the above expressions for the plasma parameters into the continuity equation for ions

$$\frac{\partial N_{\mathrm{i}}}{\partial t} + \frac{\partial}{\partial x}(N_{\mathrm{i}} w_{\mathrm{i}}) = 0$$

This yields the following relationship between the oscillation amplitudes of the ionic parameters:

$$\omega N_{\mathrm{i}}' = k N_0 w_{\mathrm{i}}' \tag{3.8 a}$$

From the motion equation for ions $M(d\mathbf{w}_i/dt) = e\mathbf{E} = -e\nabla\varphi$ (M is the ionic mass) we obtain

$$M\omega w_i' = ek\varphi' \tag{3.8 b}$$

The Poisson equation is

$$\operatorname{div}\mathbf{E} \equiv -\Delta\varphi = 4\pi e(N_i - N_e)$$

Substituting into it the above expressions for the number densities of ions and electrons and using the expressions for the spatial distribution of the electric field potential and the Boltzmann distribution for the electron number density, we obtain

$$k^2\varphi' = 4\pi eN_i' - \frac{4\pi e^2N_0}{T_e}\varphi' \tag{3.8 c}$$

Eliminating from this system of equations the oscillation amplitudes of the ion number density, mean ion velocity and electric field potential, we derive the following dispersion relation:

$$\omega^2 = \frac{k^2T_e/M}{1 + k^2r_D^2} \tag{3.9}$$

Here $r_D = \sqrt{T_e/4\pi N_e e^2}$ is the Debye-Hückel radius, and we assumed that the electron temperature is much higher than the ion temperature. This is the dispersion relation for the *ion-acoustic wave* (ion sound).

If the wavelength of the ion-acoustic wave is large compared with the Debye-Hückel radius, the dispersion relation is transformed into

$$\omega = k\sqrt{T_e/M} \tag{3.10}$$

which has the same form as the dispersion relation for the conventional acoustic wave. This analogy can be readily extended, since the interaction between charged particles in plasma is shielded at distances of the order of the Debye-Hückel radius and has a short-range character at distances of the order of the wavelength as is the case of a gas. Plasma electrons affect the field distribution and the ion-sound velocity is determined by the electron temperature.

In the other limiting case, when the ion-sound wavelength is considerably larger than the Debye-Hückel radius, the oscillations are determined by the long-range Coulomb interaction between ions. Then the dispersion relation can be written as

$$\omega^2 = 4\pi N_0e^2/M$$

This relation is similar to that for plasma oscillations but corresponds to the motion of a heavier species.

Problem 3.7. Derive the dispersion relation for the ion-acoustic waves using the kinetic equation for plasma electrons and ions. Assume that the velocity distributions of electrons and ions have the Maxwellian form, the electron temperature T_e is considerably higher than the ion temperature T_i, and the wavelength of oscillations is considerably larger than the Debye-Hückel radius for plasma.

Since ion-acoustic waves are oscillations of plasma ions, we have to take into account here the motion of ions in addition to the motion of electrons.

The kinetic equation for electrons and ions and the Poisson equation yield for the harmonically varying plasma parameters the following equations, instead of eqs. (3.4) and (3.5):

$$-i(\omega - kv_e)\,f'_e = \frac{eE'}{m}\frac{\partial f_e}{\partial v_e}, \qquad -i(\omega - kv_i)\,f'_i = -\frac{eE'}{M}\frac{\partial f_i}{\partial v_i}$$

$$ikE' = 4\pi Ne\left(\int f'_i\,dv_i - \int f'_e\,dv_e\right)$$

Here f'_e and f'_i are the parts of the distributions for electrons and ions due to plasma oscillations, E' is the amplitude of oscillations of the field strength, ω and k are the oscillation frequency and wave vector, f_e and f_i are the Maxwellian distributions for electrons and ions, v_e and v_i are the components of the velocities of electrons and ions in the direction of wave propagation, and m and M are the masses of electron and ion. Since

$$\frac{\partial f_e}{\partial v_e} = -\frac{mv_e}{T_e}f_e, \qquad \frac{\partial f_i}{\partial v_i} = -\frac{Mv_i}{T_i}f_i$$

the dispersion relation for both wave modes is

$$1 = \frac{\omega_0^2}{k^2}\left(\frac{m}{T_i}\int\frac{v_i f_i\,dv_i}{\dfrac{\omega}{k} - v_i} + \frac{m}{T_e}\int\frac{v_e f_e\,dv_e}{\dfrac{\omega}{k} - v_e}\right) \tag{3.11}$$

This dispersion relation can be applied to plasma oscillations and the ion-acoustic waves. For low-frequency waves (the ion-acoustic waves or the ion sound) we shall assume $kv_i \ll \omega \ll kv_e$. Hence, expanding and approximating the denominator of the integrands, we obtain

$$1 = -\frac{\omega_0^2}{k^2}\frac{m}{T_e} - i\pi\frac{\omega_0^2}{k^2}\frac{m}{T_e}(v_e f_e)\Big|_{v_e=\omega/k} - i\pi\frac{\omega_0^2}{k^2}\frac{m}{T_i}(v_i f_i)\Big|_{v_i=\omega/k} + \frac{\omega_0^2}{\omega^2}\frac{m}{M}$$

Since $kv_e \ll \omega_0$, we have

$$\omega^2 = k^2 \frac{T_e}{M} - i\pi \frac{\omega^3}{k} \sqrt{\frac{m}{2\pi T_e}} - i\pi \frac{\omega^3}{k} \sqrt{\frac{M}{2\pi T_i}} \exp\left(-\frac{M\omega^2}{2k^2 T_i}\right) \quad (3.12)$$

Let us find the ratio of the damping factor δ to the frequency ω, assuming $\delta \ll \omega$ and $T_i \ll T_e$. We have

$$\delta = \frac{\pi\omega^2}{2k} \sqrt{\frac{m}{2\pi T_e}} + \frac{\pi\omega^2}{2k} \sqrt{\frac{M}{2\pi T_i}} \exp\left(-\frac{M\omega^2}{2k^2 T_i}\right) \quad (3.12a)$$

Since $\omega = k\sqrt{T_e/M}$, we obtain then

$$\frac{\delta}{\omega} \sqrt{\frac{\pi m}{8M}} + \frac{1}{2} \sqrt{\frac{\pi T_e}{T_i}} \exp\left(-\frac{T_e}{2T_i}\right)$$

We see that the damping factor δ for the ion sound is small compared with its frequency only if the electron temperature is considerably higher than the ion temperature. Thus, ion sound can propagate in plasma only if the electron temperature is high compared with the ion temperature.

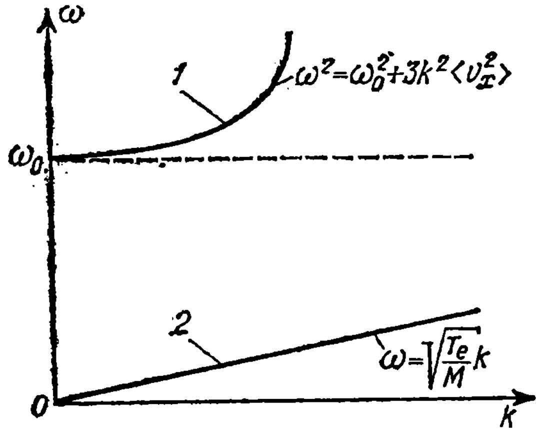

Fig. 3.1. Oscillations in a plasma in the absence of external fields. *1* — plasma oscillations; *2* — ion sound.

Thus, two types of plasma oscillations can develop in a plasma in the absence of external fields (see Fig. 3.1). The first type of oscillations, the high-frequency mode, is known as plasma oscillations and is due to the motion of plasma electrons. The low-frequency oscillations are due to the motion of plasma ions in the self-consistent field of electrons. The electrons have higher velocities and adiabatically follow the motion of ions and thus affect it. Therefore, the frequency of the ion sound depends on the parameters of the electron component of the plasma.

Problem 3.8. Derive the Debye shielding of an electric field in a plasma from the dispersion relation (3.11).

Shielding of a constant electric field penetrating plasma is due to the effect of electrons and ions, and to describe it we have to make use of the dispersion relation taking into account the motion of both electrons and ions of plasma. Hence, we shall use eq. (3.11) which was derived without assuming that the wave vector was small. The time and coordinate dependence of the wave field can be described by the relation $\mathbf{E} = \mathbf{E}_0 \exp[i(kx -$

$- \omega t)]$ which is transformed into $\mathbf{E} = \mathbf{E}_0 \exp(- \alpha x)$, where $k = i\alpha$ foı $\omega = 0$. For Maxwellian velocity distributions of ions and electrons and for $\omega = 0$ the dispersion relation (3.11) yields

$$k^2 = - m\omega_0^2 \left(\frac{1}{T_e} + \frac{1}{T_i} \right) = - r_D^{-2}$$

where the Debye-Hückel radius r_D is given by eq. (2.7). Hence, $k = i/r_D$ and $\mathbf{E} = \mathbf{E}_0 \exp(- x/r_D)$.

Problem 3.9. Derive the dispersion relation for an electromagnetic wave propagating in plasma.

Write down the Maxwell equations relating the electric $\mathbf{E}$ and magnetic $\mathbf{H}$ fields in the electromagnetic wave:

$$\operatorname{curl} \mathbf{E} = - \frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}$$

$$\operatorname{curl} \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{j}$$

Here $\mathbf{j}$ is the current density of charged plasma particles. Eliminating from the Maxwell equation the magnetic field $\mathbf{H}$, we obtain

$$\Delta \mathbf{E} - \nabla \operatorname{div} \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{4\pi \partial \mathbf{j}}{c^2 \partial t} \qquad (3.13)$$

Since the amplitude of the electromagnetic wave is small, it does not change the plasma properties. Therefore the electric field strength can be expressed as the sum of two parts $\mathbf{E}_0 + \mathbf{E}'$, where the first term is the field in the absence of electromagnetic waves and the second term is due to the wave. Then the Maxwell equation (3.13) is transformed into two independent equations; the field of the electromagnetic wave is described by eq. (3.13) if the current density in this equation is due to this wave. If we assume that the ion velocity is small compared with the electron velocity, the current density is

$$\mathbf{j} = - N_e e \mathbf{v}_e'$$

where N_e is the electron number density, and $\mathbf{v}_e'$ is the electron velocity component due to the electromagnetic wave. This velocity can be found from the motion equation for an electron $m(d\mathbf{v}_e'/dt) = - e\mathbf{E}'$, and then we can rewrite eq. (3.13) as

$$\Delta \mathbf{E}' - \operatorname{grad} \operatorname{div} \mathbf{E}' = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}'}{\partial t^2} + \frac{4\pi N_e e^2}{mc^2} \mathbf{E}'$$

For a plane wave we have $\mathbf{E}' = \mathbf{E}'_0 \exp[i(\mathbf{kr} - \omega t)]$. Substituting this expression into the above equation, we obtain the dispersion relation between the electromagnetic wave frequency ω and the wave vector $\mathbf{k}$. For a transverse electromagnetic wave we have $\mathbf{kE} = 0$, and finally we obtain

$$\omega^2 = \omega_0^2 + k^2c^2 \tag{3.14}$$

where

$$\omega_0 = (4\pi N_e e^2/m)^{\frac{1}{2}} \tag{3.15}$$

is the plasma frequency. The dispersion relation (3.14) shows that the electromagnetic waves with frequencies higher than the plasma frequency can propagate in plasma. If the number density of charged particles in a plasma is low, the dispersion relation (3.14) is identical to the dispersion relation for the electromagnetic wave in a vacuum ($\omega = kc$).

Problem 3.10. Determine the character of propagation of electromagnetic waves in a plasma if the motion of ions can be ignored.

Let us derive the dispersion relation for electromagnetic waves similar to that found in Problem 3.9 but taking into account damping due to collisions. The Maxwell equations yield eq. (3.13) for the electric field of the wave. Since the wave parameters depend on time and coordinates as $\exp[-i(\omega t + \mathbf{kr})]$, we can rewrite eq. (3.13) as

$$k^2\mathbf{E} - \mathbf{k}(\mathbf{kE}) = \frac{\omega^2}{c^2}\mathbf{E} - \frac{4\pi i\omega}{c^2}\mathbf{j} \tag{3.16 a}$$

where $\mathbf{j} = -eN_e\mathbf{v}_e$ is the electron current density, N_e is the electron number density, and $\mathbf{v}_e$ is the mean velocity of electrons due to the wave. Since the electromagnetic wave is transverse, $\mathbf{kE} = 0$ and $\operatorname{div}\mathbf{E} = 0$. This shows that plasma remains quasineutral and the perturbations under the effect of the wave result in the motion of electrons.

Under the conditions of this problem we ignore the motion of ions and assume that the electric current affecting propagation of the electromagnetic wave is due to the motion of electrons. We shall determine the electron current from the kinetic equation for electrons in the tau-approximation

$$\frac{\partial f}{\partial t} + \mathbf{v}\nabla f - \frac{e\mathbf{E}}{m}\frac{\partial f}{\partial \mathbf{v}} = -\nu(f - f_0)$$

where f_0 is the equilibrium velocity distribution of electrons in a plasma in the absence of electromagnetic waves, $\mathbf{E}$ is the electric field of the wave, and ν is the frequency of collisions between electrons and the gas particles. Write down the distribution function for electrons in the form $f = f_0 + f'$, where the term f' is due to the electromagnetic wave and its dependence of time and coordinates is taken in the usual form $f' \sim \exp[i(\mathbf{kr} - \omega t)]$.

Then the kinetic equation is reduced to

$$f'(-i\omega + i\mathbf{kr} - \nu) - \frac{e\mathbf{E}}{m}\frac{\partial f_0}{\partial \mathbf{v}} = 0$$

Taking the wave vector **k** parallel to the z-axis and the electric field **E** of the wave parallel to the x-axis, we obtain

$$f' = \frac{ieE}{m}\frac{\partial f_0}{\partial v_x}(\omega - kv_z + i\nu)^{-1}$$

According to eq. (3.16a), the electric field of the wave and the electric current density have the same direction.

Therefore, the electric current is parallel to the x-axis and its density is, by definition,

$$j_x = -eN_0\langle v_x\rangle = -\frac{ie^2E}{m}\int\frac{v_x\dfrac{\partial f_0}{\partial v_x}dv_x}{\omega - kv_z + i\nu}$$

where N_0 is the electron number density and the angle brackets denote averaging over electron velocities. Taking the integral by parts, we obtain

$$\mathbf{j} = \frac{iN_0e^2\mathbf{E}}{m}\left\langle\frac{1}{\omega - kv_z + i\nu}\right\rangle \tag{3.16 b}$$

Substituting eq. (3.16b) into eq. (3.16a), we obtain the following dispersion relation:

$$\omega^2 = k^2c^2 + \omega\omega_0^2\left\langle\frac{1}{\omega - kv_z + i\nu}\right\rangle$$

Here ω_0 is the plasma frequency (3.15).

For high frequencies this relation is transformed into the relation which is similar to that obtained in Problem 3.9 but takes into account damping:

$$\omega^2 = k^2c^2 + \omega_0^2\left(1 - \frac{i\nu}{\omega}\right)$$

In the other limiting case, when $\omega \ll k\sqrt{T_e/m}$ (T_e is the electron temperature), we find the residue of the integrand and obtain

$$0 = k^2c^2 - i\frac{\omega\omega_0^2}{k}\sqrt{\frac{\pi m}{2T_e}}$$

and

$$\omega = -i\sqrt{\frac{2T_e}{\pi m}}\frac{c^2k^3}{\omega_0^2}$$

This dispersion relation can be used to find the depth of penetration into plasma for a low-frequency transverse wave. Let the electromagnetic wave propagate along the z-axis perpendicular to the plasma boundary. Since the wave amplitude varies with coordinate as $\exp(ikz)$, the wave damping is described by $\exp(-z/\delta)$, where $1/\delta = \operatorname{Im} k$. For low-frequency waves the characteristic depth of wave damping is

$$\delta = \frac{1}{\operatorname{Im} k} = 2\left(\frac{2T_e}{\pi m}\right)^{1/6}\left(\frac{c^2}{\omega\omega_0}\right)^{1/3} \tag{3.17}$$

It can be seen that in the low-frequency limit the depth of penetration of transverse waves into plasma tends to infinity. This is known as the *anomalous skin effect* (see Problem 2.6).

Problem 3.11. Derive the dispersion relation for high-frequency waves propagating in a plasma in a constant magnetic field. The plasma is assumed to be cold, that is, the gas pressure is small compared with the magnetic pressure.

Let us write down the general system of equations for waves in a plasma in the presence of a magnetic field, using the Maxwell equations and the motion equations for charged species. Eliminating from the Maxwell equations

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}'}{\partial t}, \quad \operatorname{curl} \mathbf{H}' = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}$$

the magnetic field $\mathbf{H}'$ of the wave and taking into account the harmonic dependence of the electric field $\mathbf{E}$ and the current density $\mathbf{j}$ on time and coordinates ($\sim \exp[i(\mathbf{kr} - \omega t)]$), we can come to the following equation:

$$k^2\mathbf{E} - \mathbf{k}(\mathbf{kE}) = \frac{\omega^2}{c^2}\mathbf{E} + i\frac{4\pi\omega}{c^2}\mathbf{j} \tag{3.18 a}$$

Another equation relating $\mathbf{E}$ and $\mathbf{j}$ follows from the motion equations for the charged particles, in which we can ignore the thermal motion since the plasma is cold:

$$m\frac{d\mathbf{v}_e}{dt} = -e\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}_e\mathbf{H}]\right)$$

$$M\frac{d\mathbf{v}_i}{dt} = e\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}_i\mathbf{H}]\right)$$

Here $\mathbf{v}_e$ and $\mathbf{v}_i$ are the velocities of electrons and ions, respectively, $\mathbf{H}$ is the external magnetic field, and the magnetic field of the wave is ignored. The current density $\mathbf{j} = Ne(\mathbf{v}_i - \mathbf{v}_e)$, where N is the number density of charged particles; the velocity of the centre of mass is $\mathbf{V} = (m\mathbf{v}_e + M\mathbf{v}_i)/M$.

Since the wave parameters depend on time and coordinates as $\exp[i(\mathbf{kr} - \omega t)]$ and $M \gg m$, we can rewrite the motion equations as

$$\mathbf{V} = \frac{i}{M\omega Nc}[\mathbf{jH}]$$

$$-i\omega\mathbf{j} = \frac{\omega_0^2}{4\pi}\left(\mathbf{E} + \frac{1}{c}[\mathbf{VH}]\right) - \frac{e}{mc}[\mathbf{jH}]$$

where $\omega_0^2 = 4\pi Ne^2/m$ is the squared plasma frequency (3.15). Eliminating $\mathbf{V}$ from these equations, we obtain

$$(\omega^2 - \omega_{eH}\omega_{iH})\,\mathbf{j} + \omega_{eH}\omega_{iH}\mathbf{h}(\mathbf{jh}) + i\omega\omega_{eH}[\mathbf{jh}] - i\frac{\omega_0^2\omega}{4\pi}\mathbf{E} = 0 \quad (3.18\text{ b})$$

where $\mathbf{h}$ is the unit vector in the direction of the magnetic field $\mathbf{H}$, and $\omega_{eH} = eH/mc$ and $\omega_{iH} = eH/Mc$ are the Larmor frequencies for electrons and ions, respectively.

The system of equations (3.18a and b) can be used to derive the dispersion relations for waves in a cold plasma. For instance, in the absence of a magnetic field ($\omega_{eH} = \omega_{iH} = 0$) this system of equations is reduced to

$$k^2\mathbf{E} - \mathbf{k}(\mathbf{kE}) = \frac{\omega^2 - \omega_0^2}{c^2}\mathbf{E}$$

Hence, the dispersion relation for a longitudinal wave ($\mathbf{k} \parallel \mathbf{E}$) is $\omega = \omega_0$ and the dispersion relation for a transverse electromagnetic wave ($\mathbf{k} \perp \mathbf{E}$) is $\omega^2 = \omega_0^2 + k^2c^2$. This agrees with the results obtained in Problems 3.2 and 3.9.

Using eqs. (3.18a and b) to derive dispersion relations for high-frequency waves we shall assume $\omega_0 \gtrsim \omega_{eH}$, so that the terms with the Larmor frequency ω_{iH} for ions can be ignored. Then eq. (3.18b) is transformed into

$$\omega^2\mathbf{j} + i\omega\omega_{eH}[\mathbf{jh}] - i\frac{\omega_0^2\omega}{4\pi}\mathbf{E} = 0$$

For the waves with $\mathbf{E}$ parallel to $\mathbf{H}$ the parameters are independent of the magnetic field, since $[\mathbf{jh}] = 0$. Then we have the electromagnetic waves — the longitudinal wave ($\omega = \omega_0$) and the transverse wave ($\omega^2 = \omega_0^2 + k^2c^2$).

Let us consider the wave with $\mathbf{E}$ perpendicular to $\mathbf{H}$ and direction of propagation perpendicular to $\mathbf{H}$. Take the x-axis parallel to $\mathbf{H}$ and the y-axis parallel to the direction of the wave propagation $\mathbf{k}$. Then the system of equations (3.18a and b) can be written in terms of the vector components:

$$\omega^2 j_y + i\omega\omega_{eH}j_z - i\frac{\omega_0^2\omega}{4\pi}E_y = 0$$

$$\omega^2 j_z - i\omega\omega_{eH} j_y - i\frac{\omega_0^2\omega}{4\pi}E_z = 0$$

$$\frac{\omega^2}{c^2}E_y + i\frac{4\pi\omega}{c^2}j_y = 0$$

$$\left(\frac{\omega^2}{c^2} - k^2\right)E_z + i\frac{4\pi\omega}{c^2}j_z = 0$$

This system of equations yields the following dispersion relation:

$$(\omega^2 - \omega_0^2)(\omega^2 - k^2c^2 - \omega_0^2) - (\omega^2 - k^2c^2)\,\omega_{eH}^2 = 0 \tag{3.19}$$

This relation describes two types of waves — the longitudinal plasma waves and the transverse electromagnetic waves. If the frequencies of these waves differ greatly, they are independent of each other. This is the case when $k^2c^2 \gg \omega_0^2$; then eq. (3.19) yields $\omega^2 = k^2c^2 + \omega_0^2$ for the electromagnetic waves and $\omega^2 = \omega_0^2 + \omega_{eH}^2$ for the plasma waves.

Problem 3.12. Derive the dispersion relation for the low-frequency waves propagating in a cold plasma in a constant magnetic field.

We shall make use of the system of equations (3.18 a and b). The condition that the wave frequency is low means that $\omega \ll \omega_{iH}$. Moreover, to distinguish clearly the plasma oscillations under consideration from the electromagnetic waves, we shall assume that $kc \gg \omega$. The result will show that this condition (the velocity of wave propagation is small compared with the velocity of light) is always satisfied. Under such assumptions the system of equations (3.18 a and b) is transformed into

$$k^2\mathbf{E} - \mathbf{k}(\mathbf{kE}) = i\frac{4\pi\omega}{c^2}\mathbf{j} \tag{3.20 a}$$

$$\omega_{eH}\omega_{iH}[\mathbf{j} - \mathbf{h}(\mathbf{jh})] + i\frac{\omega_0^2\omega}{4\pi}\mathbf{E} = 0 \tag{3.20 b}$$

The system of equations (3.20 a and b) shows that there are two possible types of low-frequency waves in a plasma in a magnetic field.

1. Magnetohydrodynamic waves. These waves propagate parallel to the magnetic field (**k**||**H**) and their electric field **E** is perpendicular to the magnetic field **H**. The system of equations (3.18 a and b) yields the following dispersion relation for the magnetohydrodynamic waves:

$$\omega = kc\frac{\sqrt{\omega_{eH}\omega_{iH}}}{\omega_0} = k\frac{H}{\sqrt{4\pi MN}} \tag{3.21}$$

2. **Magnetic sound.** These transverse waves ($\mathbf{k} \perp \mathbf{E}$) propagate perpendicular to the magnetic field ($\mathbf{k} \perp \mathbf{H}$) and are polarized perpendicular to the magnetic field ($\mathbf{E} \perp \mathbf{H}$). The dispersion relation for magnetic sound is also given by eq. (3.21). Though magnetohydrodynamic waves and the magnetic sound have different physical nature for fairly large wavelengths ($\omega \ll \omega_{iH}$), their velocity of propagation is the same and they are described by the same dispersion relation (3.21).

The magnetohydrodynamic waves are often referred to as the *Alfven waves*, after the Swedish physicist H. Alfven who discovered them. The phase velocity

$$c_A = \omega/k = H/\sqrt{4\pi MN}$$

of the waves of both types (the magnetohydrodynamic waves and magnetic sound) is known as the *Alfven speed.*

Both types of waves can be described in the framework of the following simple physical model. The conditions of this problem correspond to the conditions of magnetohydrodynamics (see Problem 2.44), when the plasma conductivity is infinite and the magnetic field lines are "frozen" into plasma. Then any displacement of the magnetic field lines results in the displacement of the plasma and this displacement has a wave nature owing to the elasticity of plasma. Consider the velocity of propagation of the resulting elastic waves which is, by definition, $c = \sqrt{\partial p/\partial \rho}$, where p is the pressure, and $\rho = MN$ is the density of plasma. The pressure of a cold plasma coincides with the magnetic pressure $p = H^2/8\pi$, so that the propagation velocity of the waves is $c = \left(\frac{H}{4\pi M}\frac{\partial H}{\partial N}\right)^{1/2}$. Since the magnetic field lines are frozen into plasma, we have $\partial H/\partial N = H/N$. Thus, the velocity of propagation of these waves is equal to the Alfven speed.

Two types of these waves are possible depending on their direction of propagation. If the oscillation of the magnetic field line propagates along the field line, that is, parallel to the magnetic field, this is the Alfven wave which is similar to the wave propagating along an elastic string. The oscillation of a given magnetic field line in the plasma can give rise to oscillations of the neighbouring field lines, that is, produce a wave propagating perpendicular to the magnetic field. Such waves are the magnetic sound.

Problem 3.13. Analyze the propagation of waves in a cold magnetized plasma if the wave frequency is much lower than the electron Larmor frequency but much higher than the ion Larmor frequency.

We shall make use of the equations for wave parameters derived above to obtain the dispersion relation. The first such equation follows from the Maxwell equations and can be written in the form (3.13). Assuming that the frequency is low, $\omega \ll kc$, we can rewrite eq. (3.13) as

$$k^2\mathbf{E} = \mathbf{k}(\mathbf{kE}) - i\frac{4\pi\omega}{c^2}\mathbf{j} = 0 \tag{3.22 a}$$

where **k** is the wave vector of the wave, **E** is the electric field of the wave, ω is its frequency, c is the speed of light, and **j** is the current density. The current here is carried by electrons, so that $\mathbf{j} = -eN_0\mathbf{v}_e$, where N_0 is the electron number density, and $\mathbf{v}_e$ is the electron velocity in the wave. The motion equation for electrons is

$$m\frac{d\mathbf{v}_e}{dt} = -e\mathbf{E} - \frac{e}{c}[\mathbf{v}_e\mathbf{H}]$$

where **H** is the external magnetic field. Using the time dependence of the electron speed in the wave in the form $v_e \sim \exp[i(\mathbf{kr} - \omega t)]$ and the Larmor frequency $\omega_H = eH/mc$ for electrons, we can rewrite this motion equation as

$$-i\omega\mathbf{v}_e + \omega_H[\mathbf{v}_e\mathbf{h}] + e\mathbf{E}/m = 0$$

where **h** is the unit vector in the direction of **H**.

Since the wave frequency is small compared with the electron Larmor frequency, we can transform the above equation into

$$\mathbf{E} = \frac{4\pi\omega_H}{\omega_0^2}[\mathbf{jh}] \tag{3.22 b}$$

where $\omega_0 = (4\pi N_0 e^2/m)^{1/2}$ is the plasma frequency (3.3). Equation (3.22 b) follows directly from eq. (3.20 b) if only the last two terms are left in it in accordance with the assumptions used in this problem.

Substituting eq. (3.22 b) into eq. (3.22 a), we obtain

$$k^2[\mathbf{jh}] - \mathbf{k}(\mathbf{k}[\mathbf{jh}]) - i\frac{\omega\omega_0^2}{\omega_H c^2}\mathbf{j} = 0 \tag{3.23}$$

Take the z-axis parallel to the external magnetic field (the unit vector **h**) and the wave vector in the plane xz. Then the x and y components of this equation are

$$(k^2 - k_x^2)j_y - i\frac{\omega\omega_0^2}{\omega_H c^2}j_x = 0$$

$$-k^2 j_x - \frac{i\omega\omega_0^2}{\omega_H c^2}j_y = 0$$

The determinant of this system of equations must be zero, and we obtain the following dispersion relation:

$$\omega = \frac{\omega_H c^2}{\omega_0^2}kk_z \equiv \frac{\omega_H c^2 k^2}{\omega_0^2}\cos\theta \tag{3.24}$$

Here θ is the angle between the direction of wave propagation and the external magnetic field. Such waves were first discovered in the Earth atmosphere; they were called the *whistling atmospherics* or *whistlers.*

It can be readily seen that the frequency of the whistling atmospherics is considerably higher than the frequency of the Alfven waves and magnetic sound. Indeed, for the whistling atmospherics propagating along the magnetic field we obtain from eqs. (3.21) and (3.23) $\omega = \omega_A^2/\omega_{iH}$, where ω_A is the frequency of the Alfven wave, and ω_{iH} is the Larmor frequency for ions. Since in this problem we have $\omega \gg \omega_{iH}$, we obtain $\omega_A \gg \omega_{iH}$ and $\omega \gg \omega_A$. Thus, we obtain the series of relationships

$$\omega_{eH} \gg \omega \gg \omega_A \gg \omega_{iH}, \quad kc \gg \omega$$

making it possible to distinguish between various types of waves in a magnetized plasma, which have different nature. The whistling atmospherics are entirely due to the motion of electrons. According to eq. (3.22 b), the motion of electrons and the resulting current in the magnetized plasma gives rise to an electric field which, in its turn, gives rise to an electron current in plasma, according to eq. (3.22 a). Thus, the plasma oscillations are generated.

Problem 3.14. Determine the polarization of the whistling atmospherics propagating along the magnetic field.

Using the dispersion relation for the whistling atmospherics, we can rewrite eq. (3.23) in this case as

$$[\mathbf{j}\mathbf{h}] = i\mathbf{j}$$

or $j_x = ij_y$. Equation (3.22 b) yields for the components of the electric field of the wave $E_y = iE_x$. Thus, the wave has circular polarization.

This wave propagating along the field has a helical structure. The direction of rotation of wave polarization is the same as the direction of electron rotation. The development of such a wave can be described as follows. Let electrons in a certain region acquire a velocity perpendicular to the magnetic field. This electron motion gives rise to an electric field and makes electrons circulate in the plane perpendicular to the magnetic field. This perturbation is transferred to the neighbouring regions with a phase delay. Such a wave is known as the *helicon wave.*

Problem 3.15. Analyze the wave properties of a nonuniform plasma in an external magnetic field which is perpendicular to the direction of variation of plasma density.

Electron drift along the density gradient in the plasma gives rise to an electric field perpendicular to the magnetic field and the density gradient. This electric field produces reverse motion of electrons. These oscillations are known as the *drift waves.* Now we shall derive the dispersion relation for the drift waves.

Assume that the phase velocity is high compared with the thermal velocity of ions and small compared with the thermal velocity of electrons:

$$\sqrt{T_i/M} \ll \omega/k \ll \sqrt{T_e/m}$$

Here ω is the frequency of the drift wave, k is the wave vector, T_e and T_i are the electron and ion temperatures, and m and M are the masses of an electron and an ion. According to the inequality at the left we can ignore the motion of ions in the analysis of the drift waves. According to the inequality at the right, during propagation of the drift wave the electron distribution has the equilibrium Boltzmann form

$$N_e = N_0 \exp(e\varphi/T_e)$$

where N_e is the electron number density in the presence of the drift wave, N_0 is the electron number density in the absence of the drift wave, and φ is the potential of the electric field of the wave. For a small-amplitude wave we can write $N_e = N_0 + N'$ and obtain

$$N' = N_0 \frac{e\varphi}{T_e}$$

Let us make use of the continuity equation for electrons with the mean velocity $\mathbf{v}_e$:

$$\frac{\partial N_e}{\partial t} + \operatorname{div}(N_e \mathbf{v}_e) = 0$$

Hence

$$-i\omega N' + v_{ex} \frac{\partial N_0}{\partial x} = 0$$

where ω is the frequency of the drift wave, the equilibrium plasma density varies along the x-axis, and v_{ex} is the drift velocity of electrons under the effect of the magnetic field and the electric field of the wave:

$$v_{ex} = cE_y/H = -ick_y\varphi/H$$

Here the y-axis is perpendicular to the magnetic field and the plasma density gradient.

Substitute the above expression for the drift velocity of electrons into the continuity equation for electrons and eliminate from the result the electron density and the electric field potential due to the oscillations by making use of the Boltzmann distribution for electrons. We obtain then the following dispersion relation for the drift waves:

$$\omega = -k_y \frac{cT_e}{eH} \frac{d(\ln N_0)}{dx} \tag{3.25}$$

The condition of this problem is that the phase velocity of the wave is much lower than the thermal velocity of electrons and much higher than the thermal velocity of ions. Making use of the above dispersion relation, we can rewrite this condition as

$$1 \ll \frac{\omega_H L}{\sqrt{T_e/m}} \ll \sqrt{\frac{M}{m}\frac{T_e}{T}}$$

where $\omega_H = eH/mc$ is the Larmor frequency for electrons, and L is the characteristic distance over which the plasma density changes noticeably.

Problem 3.16. Derive the dispersion relation for waves propagating in a plasma, making use of the dielectric constant of plasma.

The approach to deriving the dispersion relation between the wave frequency and the wave vector used in solving the problems above can be generalized as follows. Write down linearized equations describing the wave processes. This means that the amplitudes of variation of the plasma parameters in the wave are assumed to be small compared with the characteristic values of these parameters in the plasma. Then use the condition of existence of a nonzero solution of the resulting system of linear equation, that is, set the determinant of this system equal to zero. The same relation can be obtained by eliminating all the parameters but one from this system of equations and taking the coefficient at the sought parameter equal to zero in the remaining single equation. This is also a condition of existence of the nonzero solution.

This procedure can be generalized by describing the relationships between the electric and magnetic parameters in a plasma by the Maxwell equations and the wave processes by the dielectric constant of the plasma. Then we obtain eq. (3.18 a) which can be written in terms of vector components as

$$k^2 E_\alpha - k_\alpha k_\beta E_\beta - \frac{\omega^2}{c^2} E_\alpha - i\frac{4\pi\omega}{c^2} j_\alpha = 0$$

where summation is implied over the repeated subscripts. Further, using Ohm's law $j_\alpha = \Sigma_{\alpha\beta} E_\beta$ (where $\Sigma_{\alpha\beta}$ is the plasma conductivity) and the relationships (2.41) between conductivity and dielectric constant, we obtain

$$k^2 E_\alpha - k_\alpha k_\beta E_\beta - \frac{\omega^2}{c^2} \varepsilon_{\alpha\beta} E_\beta = 0$$

The nonzero solution of this system of equations for the electric field components exists if the determinant of this system is zero:

$$\left| k^2 \delta_{\alpha\beta} - k_\alpha k_\beta - \frac{\omega^2}{c^2} \varepsilon_{\alpha\beta} \right| = 0$$

This is the dispersion relation for the waves in a plasma, in which the properties of the waves are described by the function $\varepsilon_{\alpha\beta}(\omega, k)$.

The simplest case is that of an isotropic plasma for which the dielectric constant tensor $\varepsilon_{\alpha\beta}$ has a diagonal form. In this case we deal with the longitudinal and transverse waves, that is, the waves whose electric field is parallel or perpendicular to the direction of propagation. The dispersion relation for the longitudinal waves is

$$\varepsilon_{\parallel}(\omega, k) = 0 \tag{3.25 a}$$

and the relation for the transverse waves is

$$c^2k^2/\omega^2 - \varepsilon_{\perp}(\omega, k) = 0 \tag{3.25 b}$$

where $\varepsilon_{\parallel}$ and $\varepsilon_{\perp}$ are the dielectric constant in the direction of wave propagation and perpendicular to it.

In particular, for a cold plasma ($kv_e \ll \omega$, v_e is the characteristic electron velocity) eq. (2.43) yields $\varepsilon = 1 - \omega_0^2/\omega^2$, where ω_0 is the frequency of plasma oscillations. Hence eq. (3.25 a) yields $\omega = \omega_0$ for the longitudinal plasma oscillations with $k = 0$ according to eq. (3.7). For the transverse waves (electromagnetic oscillations) eq. (3.25 b) yields $\omega^2 = \omega_0^2 + k^2c^2$, which coincides with eq. (3.14).

The dispersion relations (3.25a and b) employing the dielectric constant are convenient for the analysis because individual plasma components make independent contributions to the dielectric constant of the plasma. Thus, contributions of individual groups of ions or electrons to oscillations in plasma can be taken into consideration separately.

§ 3.2. ATTENUATION AND GROWTH OF WAVES IN PLASMA

Interactions between oscillations in a plasma and the plasma itself or the external fields applied to the plasma result in the energy exchange between oscillations of plasma and its other degrees of freedom or the external fields. If the waves lose energy, they decay, and if the waves absorb energy they are intensified. In the former case the appearance of oscillations in the plasma has no significant effect on its parameters, since random small-amplitude oscillations in the plasma do not develop further. In the latter case small-amplitude oscillations will intensify and their amplitude can become so high that the plasma parameters will be changed. Such plasma states are known as *unstable states* and the developing plasma oscillations are called *instabilities*.

There are various types of instabilities. One instability type is due to the difference in the character of interaction between the groups of charged particles and waves. The instabilities which ultimately lead to a change in the velocity distribution of charged particles are known as *kinetic instabilities*. An example of a kinetic instability is the beam instability which develops when a beam of electrons passes through a plasma. The electrons of the beam produce a resonance effect on plasma oscillations and intensify them while decelerating.

Other instability types lead to spatial redistribution of charged particles in a plasma. The fastest growing instabilities of this type are the *hydrodynamic instabilities* which develop if the spatial distribution of the plasma is unstable with respect to small displacements. The development of such instabilities results in a breakdown of plasma configuration. A slower-growing type of instability is the *dissipative*, or *overheating*, *instability* which is due to the increased liberation of energy with small perturbations in some plasma regions, resulting in the growth of the perturbations. Drift instabilities due to charged particles leaving a plasma develop even slower.

The development of instabilities leads to the wave amplitudes growing to finite amplitudes determined by their interactions with particles and other types of waves in a plasma. As a result, a wide variety of waves and noises can be generated in the plasma; their interaction with the plasma leads to irregular behaviour of electric fields in the plasma and other plasma parameters. This state of a plasma is referred to as *turbulence*. The energy associated with the waves and noises of a turbulent plasma can be comparable to the thermal energy of the plasma.

To analyze the phenomena involved in the appearance and development of a plasma turbulence, we have to study the interactions of finite-amplitude waves with each other and with plasma. These interactions give rise to a variety of peculiar and fascinating effects. One such effect is the modulation instability due to the breaking of a wave packet, propagating in a plasma, into solitary waves, or *solitons*.

Problem 3.17. Analyze the decay of plasma waves in a collisionless plasma.

Write down the frequency of plasma oscillations in the form $\omega = \mathrm{Re}\,\omega - i\delta$. According to eq. (3.7), for the damping coefficient for plasma waves $[(\omega - \omega_0)/\omega_0 \ll 1]$ we have

$$\delta = -\frac{\pi\omega_0^3}{2k^2}\frac{\partial f_0}{\partial v_x}\bigg|_{v_x=\omega/k} \tag{3.26}$$

If $\dfrac{\partial f_0}{\partial v_x}\bigg|_{v_x=\omega/k}$ is negative, then the sign of δ corresponds to damping of the wave. This type of damping is known as the *Landau damping*. For instance, if the distribution function has the Maxwellian form, we have

$$\delta = \frac{\pi}{2}\frac{\omega_0^3}{k^2}\sqrt{\frac{m}{2\pi T}}\exp\left(-\frac{m\omega_0^2}{2k^2T}-\frac{3}{2}\right) \tag{3.27}$$

This shows that for $kr_D \sim \frac{k}{\omega_0}\sqrt{\frac{T}{m}} \ll 1$ the damping coefficient is much lower than the frequency of the plasma waves.

If $\left.\frac{\partial f_0}{\partial v_x}\right|_{v_x=\omega/k}$ is positive, the damping coefficient is negative and the wave is intensified. The wave receives the energy from electrons and increases until nonlinear effects due to the interaction between the wave and the plasma particles become significant. Thus, if the derivative of the distribution function is positive, this electron distribution is unstable with respect to plasma waves. Such a situation is realized when an electron beam is passed through a plasma. Even in the absence of collisions with the gas particles the electrons decelerate giving their energy to rapidly growing plasma waves. This type of instability is called the *beam instability*.

Let us describe the physical picture of damping and growth of oscillations. We shall use the reference frame in which the wave is at rest. The electrons whose energy in this reference frame is lower than the oscillation amplitude U_0 (see Fig. 3.2) are trapped in the potential well. They remain in this well receiving and returning the energy to the wave until they are withdrawn from the well owing to the energy exchange with other electrons.

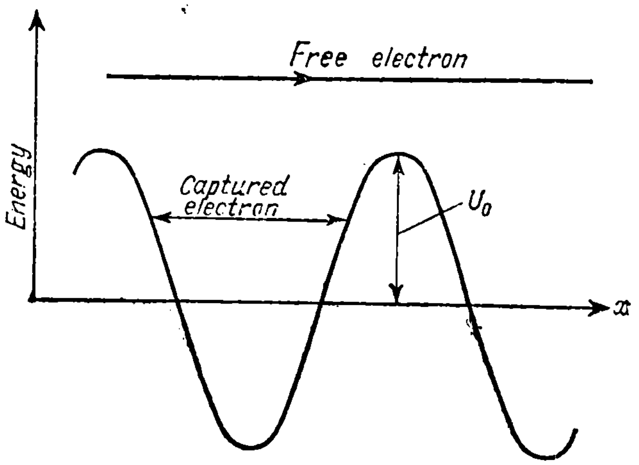

Fig. 3.2. Interaction of plasma waves with electrons. The reference frame travels with the wave.

If the velocity of the electron captured by the wave is somewhat higher than the phase velocity of the wave, it gives the energy to the wave, being reflected from the walls of the potential well.

Indeed, let an electron with the velocity $v_{ph} + u$ be reflected from the wall of the potential well. Elastic scattering by the wall results in the electron velocity being changed from u to $-u$ in the frame of reference moving with the wave. In the laboratory frame of reference the variation of the electron energy is

$$\frac{1}{2}m(v_{\text{ph}}-u)^2-\frac{1}{2}m(v_{\text{ph}}+u)^2=-2mv_{\text{ph}}u$$

and this energy is transferred to the wave. The captured electrons whose velocity is lower than the phase velocity of the wave receive the energy from the wave. To determine the direction of energy transfer between the wave and the electrons in total, we shall integrate the energy variation over all the captured electrons. We obtain

$$\int_{-u_0}^{u_0} f(v_{\text{ph}}+u)\,\Delta\varepsilon\,du=\left.\frac{\partial f}{\partial v}\right|_{v=v_{\text{ph}}}\int_{-u_0}^{u} u\,\Delta\varepsilon\,du,\qquad u_0=\sqrt{\frac{2eU_0}{m}}$$

since the variation of the electron energy $\Delta\varepsilon=-2mv_{\text{ph}}u$ due to the interaction with the wave is an odd function of the velocity u.

Thus, the direction of energy transfer between electrons and the wave is determined by the sign of the derivative $\left.\frac{\partial f}{\partial v_x}\right|_{v_{\text{ph}}}$. If the derivative is negative, that is, the number of slow electrons is greater than the number of fast electrons, electrons drain energy from the wave and it decays. When the derivative is positive, electrons transfer energy to the wave, its amplitude grows and instability develops. In this simple treatment we could not find the damping coefficient since we did not take into consideration the energy exchange between an electron and the wave when electron was in the potential well. However, this simple approach can be used to determine a criterion of applicability of eq. (3.26), which is difficult to do otherwise.

When an electron is on the right-hand side of the potential well, it transfers its energy to the wave since its velocity component along the wave decreases with time. But when the electron is on the left-hand side of the potential well, it takes the energy from the wave. If there were no collisions between electrons, the captured electron would oscillate in the well for an infinitely long time with periodically varying velocity, exchanging the energy with the wave. However, when deriving eq. (3.26) we assumed that the velocity distribution function for electrons is independent of the wave. This means that the frequency ν of reaching the electron velocity equilibrium is much higher than the frequency ω of oscillations of an electron in the potential well ($\omega \ll \nu$). The electron oscillation frequency

$$\omega \sim u_0 k \sim \sqrt{eU_0/mk} \sim \sqrt{eE'k/m}$$

where $E' \sim U_0 k$ is the electric field of the wave, and k is the wave vector of the wave. Hence,

$$eE' \ll m\nu^2/k$$

where v is the frequency of collisions of electrons with electrons or other gas particles, resulting in the transfer of energy of the order of eE'/k.

Problem 3.18. Find the damping coefficient of plasma oscillations owing to collisions between electrons and the plasma particles, using the tau-approximation (see Problem 1.19).

Write the dispersion relation for the longitudinal waves in the form

$$\varepsilon(\omega, k) = 0$$

Indeed, appearance of random fields in the plasma results in the displacement of charged particles compensating the effect of these fields. Therefore, the electric induction corresponding to the natural oscillations of the plasma is zero. For an isotropic plasma this means that the dielectric constant is zero.

We shall use eq. (2.42 b) for the dielectric constant of plasma. Then, in the limit $k \to 0$ the dispersion relation yields

$$1 = \frac{\omega_0^2}{\omega\left(\omega + \frac{i}{\tau}\right)}$$

Since $\omega_0\tau \gg 1$, we can further write

$$\omega^2 = \omega_0^2 - i\frac{\omega_0}{\tau}, \qquad \omega = \omega_0 - \frac{i}{2\tau}$$

It can be seen that the role of collisions is taken into account by adding the term $-i\frac{\omega_0}{\tau}$ to the right-hand side of the dispersion relation (3.7) for plasma oscillations.

Thus, collisions of electrons with the plasma particles result in damping of plasma waves with the damping factor $1/2\tau$. The plasma oscillations exist if $\omega_0\tau \gg 1$.

Problem 3.19. Show that electron-electron collisions in an ideal plasma cannot produce a damping coefficient comparable to the frequency of plasma oscillations.

The damping coefficient of plasma oscillations due to electron-electron collisions is of the order of the frequency of such collisions: $\delta \sim Nv\sigma$, where N is the number density of electrons, $v \sim \sqrt{T/m}$ is the characteristic velocity of the collisions, and $\sigma \sim e^4/T^2$ is the cross section of collision between electrons due to the Coulomb interaction between them. A comparison between the collision frequency and the frequency of plasma oscillations yields

$$\frac{\delta}{\omega_0} \sim \frac{N\sqrt{T/m}\,e^4/T^2}{\omega_0} \sim \left(\frac{e^6 N}{T^3}\right)^{1/2} \ll 1$$

Problem 3.20. Derive the damping coefficient for plasma waves, using the motion equation for an electron in the field of the wave.

The motion equation for an electron in the field of the wave is

$$m\frac{dv_x}{dt} = -eE\cos(kx - \omega t)$$

where E is the electric field of the wave. Assuming that the wave amplitude is small, we can expand all quantities in terms of a small parameter proportional to E. The electron velocity can be written as $v_x = v_0 + v_1 + v_2 + \ldots$, where $v_n \sim E^n$. Then the motion equation yields

$$m\frac{dv_0}{dt} = 0, \quad \frac{mdv_1}{dt} = -eE\cos(kx_0 - \omega t)$$

$$m\frac{dv_2}{dt} = -eE\{\cos[k(x_0 + x_1) - \omega t] - \cos(kx_0 - \omega t)\}$$

$$= eEkx_1\sin(kx_0 - \omega t) = eEk\int_0^t \sin(kx_0 - \omega t)v_1\,dt$$

where the electron coordinate $x = x_0 + x_1 + \ldots$, x_0 is the zero approximation, $x_0 + x_1$ is the first approximation, and so on. When $t = 0$ we have $v_0 = \text{const}$ and $v_1 = v_2 = \ldots = v_n = 0$.

Correct to the second-order terms in E, the variation of the electron energy per unit time is

$$\frac{d\varepsilon}{dt} = mv_x\frac{dv_x}{dt} = mv_0\frac{dv_1}{dt} + mv_1\frac{dv_1}{dt} + mv_0\frac{dv_2}{dt}$$

Let us solve the motion equations for the electron to find $d\varepsilon/dt$. In the zero approximation we have $x_0 = v_0 t + a$, where a is the electron coordinate at the starting moment. Hence, in the first approximation we obtain

$$v_1 = -\frac{eE}{m}\,\frac{\sin[(kv_0 - \omega)t + ak] - \sin ak}{kv_0 - \omega}$$

and

$$x_1 = \int_0^t v_1\,dt = -\frac{eE}{m}\left\{\frac{\cos ka - \cos[ka + (kv_0 - \omega)t]}{(kv_0 - \omega)^2} - \frac{t\sin ka}{kv_0 - \omega}\right\}$$

Let us average $\overline{d\varepsilon/dt}$ over the initial coordinate a of the electron. Since $\overline{\sin ka} = \overline{\cos ka} = 0$ and $\overline{\sin^2 ka} = \overline{\cos^2 ka} = 1/2$, we obtain

$$\frac{\overline{d\varepsilon}}{dt} = \frac{e^2E^2}{2m}\left[\frac{\sin(kv_0-\omega)\,t}{kv_0-\omega} + \frac{v_0\,k\sin(kv_0-\omega)t}{(kv_0-\omega)^2} + \frac{v_0kt\cos(kv_0-\omega)\,t}{kv_0-\omega}\right]$$

Average this expression over electron velocities. The result is the variation of the electron energy per unit time per unit volume:

$$\frac{dW_e}{dt} = \frac{e^2E^2N_e}{2m}\int f(v_0)\left[\frac{\sin(kv_0-\omega)\,t}{kv_0-\omega} + \frac{v_0k\sin(kv_0-\omega)\,t}{(kv_0-\omega)^2} + \right.$$
$$\left. + \frac{v_0kt\cos(kv_0-\omega)\,t}{kv_0-\omega}\right]dv_0$$

Here the velocity distribution function for electrons is normalized to unity and the principal value of the integral is taken. If we tend t to zero, the first term tends to zero. The integrands of the last two terms have singularities at $v_0 = \omega/k$, so that we have first to integrate them over velocities and then tend t to zero. Since we take the principal value of the integral $\int \frac{f(v_x)\,dv_x}{kv_x-\omega}$, it is finite. Therefore, when $t \to 0$ the last term tends to zero and does not make a contribution to dW_e/dt. To evaluate the integral of the second term, expand the distribution function $f(v_0)$ in the integrand in powers of $v_0 - \omega/k$. The integral of the first term of the series is zero, since the integrand is odd; the integral of the second term is

$$\frac{dW_e}{dt} = \frac{e^2E^2}{2m}N_e\left.\frac{\partial f}{\partial v_x}\right|_{v_x=\omega/k}\frac{\omega}{k}\lim_{t\to 0}\int_{-\infty}^{+\infty}dv_0\frac{\sin(kv_0-\omega)\,t}{kv_0-\omega}$$
$$= \frac{N_e e^2E^2\omega}{2mk^2}\left.\frac{\partial f}{\partial v_x}\right|_{v_x=\omega/k}\int_{-\infty}^{+\infty}dz\,\frac{\sin z}{z} = \frac{\pi e^2E^2\omega N_e}{2mk^2}\left.\frac{\partial f}{\partial v_x}\right|_{v_x=\omega/k}$$

The integrals of other terms are zero for $t \to 0$.

The variation of the energy density W_w of the wave per unit time is $dW_w/dt = -dW_e/dt$, where $W_w = \frac{E^2}{4\pi}\overline{\cos^2(kx-\omega t)} = E^2/8\pi$. We have

$$\frac{dW_w}{dt} = -2\delta W_w = -\frac{4\pi^2\omega e^2}{mk^2}N_e\left.\frac{\partial f}{\partial v_x}\right|_{\omega/k}W_w$$

and hence

$$\delta = -\frac{2\pi^2\omega e^2N_e}{mk^2}\left.\frac{\partial f}{\partial v_x}\right|_{\omega/k} = -\frac{\pi\omega_0^3}{2k^2}\left.\frac{\partial f}{\partial v_x}\right|_{\omega/k}$$

since the wave frequency $\omega \approx \omega_0$. In this approximation we obtain the result identical to eq. (3.26).

Problem 3.21. Analyze generation of plasma oscillations when a low-intensity monochromatic electron beam passes through a plasma.

Let the velocity of electrons in the beam be u, the number density of plasma electrons N_0, the number density of electrons in the beam N_b, and the plasma temperature zero. We have to derive the dispersion relation for plasma oscillations and find the increment of these oscillations.

As before, assume that the amplitude of oscillations is small and express the parameters of the plasma and the beam as the sum of the term found in the absence of oscillations and a small harmonic term due to oscillations. The equations of continuity and motion yield for the plasma electrons

$$-i\omega N_e' + ikN_0 w_e' = 0, \qquad -i\omega m w_e' = -eE'$$

where N_e' and w_e' are the number density of plasma electrons and their velocity due to oscillations, and E' is the electric field due to oscillations. Eliminating the electron velocity from these equations, we obtain

$$N_e' = -\frac{ikeE'}{m\omega^2} N_0$$

Similar equations can be written for the beam electrons:

$$-i\omega N_b' + ikuN_b' + ikN_b w_b' = 0, \qquad -i\omega m w_b' = -eE'$$

where w_b' and N_b' are the velocity and the number density of the beam electrons due to oscillations. Eliminating w_b' from these equations, we obtain

$$N_b' = -\frac{ikeE'}{(\omega - ku)^2} N_b$$

Further, write down the Poisson equation in the following form:

$$ikE' = -4\pi e (N_e' + N_b')$$

Using the above expressions for N_e' and N_b' and cancelling out the electric field, we obtain the following dispersion relation:

$$1 = \frac{\omega_0^2}{\omega^2} + \frac{\omega_0^2 N_b/N_0}{(\omega - ku)^2} \tag{3.28}$$

Here $\omega_0 = (4\pi N_0 e^2/m)^{1/2}$ is the frequency of plasma oscillations for the plasma electrons.

In the absence of the beam ($N_b = 0$) the dispersion relation is reduced to $\omega = \omega_0$, that is, to the form of eq. (3.7) for the zero plasma temperature.

The beam interacts most strongly with the waves whose phase velocity ω/k is close to the beam velocity. Moreover, the frequency of plasma waves is close to the plasma frequency. Therefore, we shall consider below the waves with the frequencies

$$\omega = \omega_0 + \delta$$

The highest value of the increment which is the imaginary part of δ corresponds to the waves whose wave vector $k = \omega_0/u$. Substituting the above expression with this wave vector into the dispersion relation and expanding the result in powers of the parameter δ/ω_0, we obtain

$$\delta = \omega_0 \left(\frac{N_b}{2N_0}\right)^{1/3} \exp\,(2i\pi n/3)$$

where n is an integer. Since $\left|\dfrac{\delta}{\omega_0}\right| = \left(\dfrac{N_b}{2N_0}\right)^{1/3} \ll 1$, the above approximation is valid.

Analyze the obtained result. The dependence of the electron number density on time and coordinate has the following form:

$$N = N_0 + N_e' \exp\,[-i(\omega t + kx)]$$

so that if the imaginary part of the frequency, which coincides with the imaginary part of δ, is positive, the wave grows. The maximum value of the growth increment for the plasma waves corresponding to $n = 1$ is

$$\operatorname{Im}\omega = \operatorname{Im}\delta = \frac{\sqrt{3}}{2}\left(\frac{N_b}{2N_0}\right)^{1/3} \omega_0 = 0.69 \left(\frac{N_b}{N_0}\right)^{1/3} \omega_0$$

The inverse parameter describes the build-up time for the plasma oscillations, that is, the time during which the oscillation amplitude increases e times. If this time is short compared with the interval between collisions of the beam electrons and the plasma particles, then deceleration of the electron beam in the plasma can be due to plasma oscillations.

Problem 3.22. Find the threshold of instability for plasma oscillations generated by an electron beam.

Let us analyze the dispersion relation (3.28) for the plasma oscillations generated by a monochromatic electron beam. The first part of this relation is plotted in Fig. 3.3. Let us consider the limiting cases depending on the ratio ku/ω (the wave vector k is determined by the geometry of the system containing the plasma). As always, we shall assume that the number density N_b of electrons in the beam is much lower than the number density N_0 of electrons in the plasma ($N_b \ll N_0$).

When $ku \gg \omega_0$, the dispersion relation (3.28) has two independent solutions at $\omega = 0$ and $\omega = ku$:

$$\omega = \left(ku \pm \omega_0 \sqrt{\frac{N_b}{N_0}}\right)\left[1 - \frac{\omega_0^2}{2(ku)^2}\right]$$

and

$$\omega = \pm\omega_0\left[1 - \frac{\omega_0^2}{2(ku)^2}\frac{N_b}{N_0}\right]$$

Significantly, all the four solutions of eq. (3.28) are real and no damping takes place. In Fig. 3.3 the ordinate value of unity corresponds in this case to the straight line *1*.

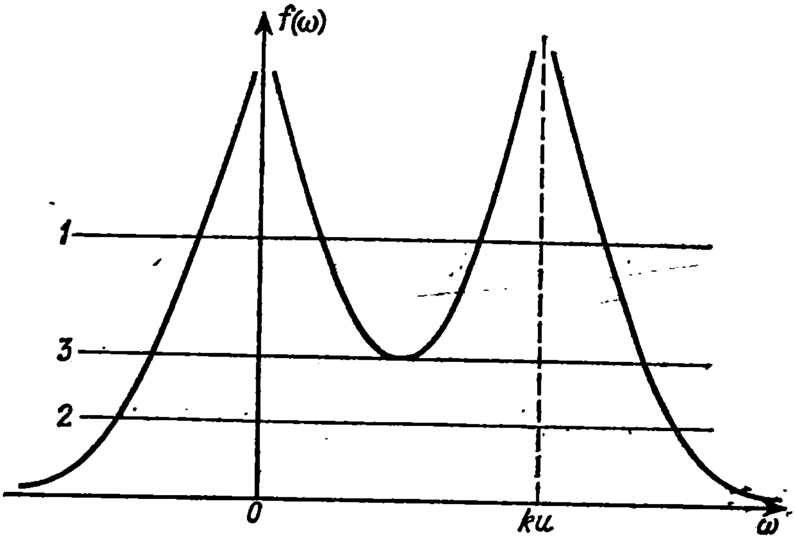

Fig. 3.3. The right-hand side of the dispersion relation (3.28) as a function of frequency.

When $ku \ll \omega_0$, the frequency is close to ku and we can ignore unity in eq. (3.28) in comparison with the term ω_0^2/ω^2. Hence, we obtain

$$\omega = ku\left(1 \pm i\sqrt{\frac{N_b}{N_0}}\right)$$

Two other solutions correspond to $\omega \gg ku$. In this case we can ignore the second term on the right-hand side of eq. (3.28). The resulting solutions are

$$\omega = \pm\omega_0$$

Since the first two solutions are imaginary, they cannot be obtained from Fig. 3.3 where only the last two solutions are shown for this limiting case. The ordinate value of unity corresponds to straight line *2*.

Note that in the second case the solutions are complex, so that one of them corresponds to growth of the oscillations under the effect of the electron beam. (The phase velocity ω/k of the oscillations is identical to the velocity of the beam electrons.) Clearly the instability threshold corresponds to the transition from one limiting case to the other. In Fig. 3.3 this is represented by the unit ordinate tangential to the inner curve of the plot of eq. (3.28) (straight line *3*).

The solution of the dispersion relation (3.28) at the instability threshold (line *3* in Fig. 3.3) is

$$\omega = \omega_0 \sqrt{1+\left(\frac{N_b}{N_0}\right)^{1/3}} = \frac{ku}{1+\left(\frac{N_b}{N_0}\right)^{1/3}}$$

Hence we can find the relationship between the parameters of the system, at which instability can develop:

$$\frac{ku}{\omega_0} \leqslant \left[1+\left(\frac{N_b}{N_0}\right)^{1/3}\right]^{3/2} \approx 1+\frac{3}{2}\left(\frac{N_b}{N_0}\right)^{1/3}$$

Under these conditions in a plasma there are oscillations whose phase velocity ω/k is identical to the velocity of the beam electrons and which, therefore, can be readily intensified. As discussed in Problem 3.31, the energy of the electron beam in this case is converted into the energy of plasma oscillations and this effect occurs at the instability threshold.

Problem 3.23. The velocity of beam electrons relative to ions in a plasma is u. Find the instability threshold due to the growth of plasma oscillations. Assume that the velocity of the beam electrons is much higher than the thermal velocity of electrons.

In this case the dispersion relation is given by eq. (3.28) in which the plasma frequency for electrons in the first term is replaced by the plasma frequency for ions and the number density of electrons in the beam taken to be equal to the total number density of electrons:

$$1 = \frac{\omega_+^2}{\omega^2} + \frac{\omega_0^2}{(\omega - ku)^2}$$

Here $\omega_+^2 = \omega_0^2 \frac{m}{M}$ is the squared plasma frequency for ions, ω_0 is the plasma frequency for electrons, and m and M are the masses of electron and ion, respectively.

Since this dispersion relation is equivalent to eq. (3.28), we can apply to it the treatment of Problem 3.22.

Rewrite the dispersion relation in the form

$$1 = \frac{\omega_0^2}{(\omega - ku)^2} + \frac{\frac{m}{M}\omega_0^2}{\omega^2}$$

According to the results of Problem 3.22, the instability threshold is at $ku \approx \omega_0$. Let $ku = \omega_0 + \Delta\omega$, where $\Delta\omega \ll \omega_0$. Moreover, $\omega \ll \omega_0$ and, hence, we can rewrite the dispersion relation retaining on the right-hand side only the first two terms of the expansion of the first term in a power series in ω/ω_0:

$$1 = \frac{\omega_0^2}{(ku)^2} + \frac{2\omega\omega_0}{(ku)^3} + \frac{\omega_0^2 \frac{m}{M}}{\omega^2}$$

Since $ku - \omega_0 = \Delta\omega \ll \omega_0$, we obtain

$$-2\frac{\Delta\omega}{\omega_0} + 2\frac{\omega}{\omega_0} + \frac{\omega_0^2}{\omega^2}\frac{m}{M} = 0$$

This equation can be conveniently rewritten as

$$f(\omega) \equiv \omega^3 - \omega^2\Delta\omega + \frac{\omega_0^3}{2}\frac{m}{M} = 0$$

The instability threshold corresponds to the solution of this equation when simultaneously $f'(\omega) = 0$. At this point the transition occurs from three real solutions to one real solution and two imaginary solutions. Solving these equations, we obtain

$$\omega = \frac{2}{3}\Delta\omega, \quad \Delta\omega = \frac{3}{2}\omega_0\left(\frac{m}{M}\right)^{1/3}$$

Thus, the instability threshold corresponds to the following relationship between the system parameters:

$$ku \leqslant \omega_0\left[1 + \frac{2}{3}\left(\frac{m}{M}\right)^{1/3}\right]$$

This instability of the electron current relative to ions is known as the *Buneman instability*. These results are identical to the results of Problem 3.22 if we replace $\omega - ku$ by ω and N_b/N_0 by m/M.

The Buneman instability is due to the growth of oscillations of ions owing to the interaction between ions and the electron beam. We assume here that the difference in electron velocities is small compared with the electron beam velocity. Note that the ion oscillations considered here differ from the ion-acoustic waves (3.9).

Problem 3.24. Analyze the decay of whistling atmospherics (whistlers) in a weakly ionized gas, taking into account the collisions between electrons and the gas particles (see Problem 3.13).

The whistlers are the oscillations whose frequency is much lower than the Larmor frequency for electrons but much higher than that for ions. The dispersion relation for the whistlers was derived in Problem 3.13. We shall derive the dispersion relation now taking into account the collisions of electrons and the gas particles. The respective equation of motion for electrons should take into account the drag force due to collisions between electrons and the gas particles. This equation can be written in the following form:

$$m\frac{d\mathbf{w}_e}{dt} = -e\mathbf{E} - \frac{e}{c}[\mathbf{w}_e\mathbf{H}] - m\mathbf{w}_e\nu$$

Here $\mathbf{w}_e$ is the mean velocity of electrons, **E** is the electric field of the wave, **H** is the external magnetic field, ν is the frequency of collisions of electrons and the gas particles, and the last term on the right-hand side of the equation is the drag force acting on electrons owing to collisions with the gas particles.

Using the procedures employed in Problem 3.13, we can transform the motion equation for electron into

$$(-i\omega + \nu)\mathbf{j} + \omega_H[\mathbf{jh}] - \frac{\omega_0^2}{4\pi}\mathbf{E} = 0 \qquad (3.29)$$

where $\mathbf{j} = -eN_0\mathbf{w}_e$ is the electron current density in the wave. We can ignore the first term in this equation since $\omega \ll \omega_H$. Substituting the resulting expression for **E** into eq. (3.22a), we obtain instead of eq. (3.23)

$$\nu k^2\mathbf{j} - \nu\mathbf{k}(\mathbf{kj}) + \omega_H k^2[\mathbf{jh}] - \omega_H\mathbf{k}(\mathbf{k}[\mathbf{jh}]) - i\frac{\omega\omega_0^2}{c^2}\mathbf{j} = 0$$

Using the reference frame employed in Problem 3.13, we can write the respective equations in terms of vector components:

$$\left(\nu k_z^2 - i\frac{\omega\omega_0^2}{c^2}\right)j_x + \omega_H k_z^2 j_y - \nu k_x k_z j_z = 0$$

$$-\omega_H k^2 j_x + \left(\nu k^2 - i\frac{\omega\omega_0^2}{c^2}\right)j_y = 0$$

$$-\nu k_x k_z j_x - \omega_H k_x k_z j_y + \left(\nu k_x^2 - i\frac{\omega\omega_0^2}{c^2}\right)j_z = 0$$

Equating the determinant of the above system of equations to zero and taking into account that $\nu \ll \omega_H$ (weak damping), we obtain finally

$$\omega = \frac{\omega_H}{\omega_0^2}c^2k^2\cos\theta\left(1 - i\frac{\nu}{\omega_H\cos\theta}\right) \qquad (3.30)$$

where θ is the angle between **H** and **k**.

It can be seen that damping of the wave is weaker for the magnetized electrons when the frequency of collisions of electrons and the gas particles is much lower than the Larmor frequency for electrons ($\nu \ll \omega_H$). Therefore, these waves can exist even if their frequency is considerably lower than the frequency of collisions of electrons and the gas particles.

Problem 3.25. Analyze the propagation and damping of cyclotron waves in a cold magnetized plasma. These waves propagate in the direction of the magnetic field and their frequency is close to the Larmor frequency, that is, the frequency of rotation of electrons in a constant magnetic field.

Let us derive the dispersion relation for the transverse waves in a cold plasma, which propagate in the direction of the magnetic field and are due to the motion of electrons. Assume that these waves are generated by the rotation of electrons in the plane perpendicular to the magnetic field. Then these waves are polarized in the plane perpendicular to the constant magnetic field, that is, the vectors of the electric field of the wave and the electron velocity are in this plane.

Let us make use of two equations derived above. The first one is eq. (3.16a) derived from the Maxwell equations for the wave fields. Since the wave vector **k** is perpendicular to the electric field **E** of the wave, we can rewrite this equation as

$$(\omega^2 - k^2c^2)\,\mathbf{E} + 4\pi i\omega\mathbf{j} = 0 \tag{3.31 a}$$

where the electron current density $\mathbf{j} = -eN_0\mathbf{v}_e$, N_0 is the electron number density, and $\mathbf{v}_e$ is the mean electron velocity.

Next, write down the motion equation for electrons assuming that ions are stationary [see eq. (3.29)]:

$$(-i\omega + \nu)\mathbf{j} + \omega_H[\mathbf{jh}] - \frac{\omega_0^2}{4\pi}\mathbf{E} = 0 \tag{3.31 b}$$

Here ν is the frequency of collisions between electrons and the gas particles, ω_H is the Larmor frequency for electrons, ω_0 is the plasma frequency, and **h** is the unit vector in the direction of the magnetic field. Substituting the expression for the electron current density derived from eq. (3.31a) into eq. (3.31b), we obtain

$$[(-i\omega + \nu)(\omega^2 - k^2c^2) + i\omega\omega_0^2]\,\mathbf{E} + (\omega^2 - k^2c^2)\omega_H[\mathbf{Eh}] = 0$$

Writing down the respective equations in vector components and equating the determinant of this system of equations to zero, we obtain the following dispersion relation:

$$[\omega(\omega^2 - k^2c^2 - \omega_0^2) + i\nu(\omega^2 - k^2c^2)]^2 - (\omega^2 - k^2c^2)^2\omega_H^2 = 0$$

We shall assume that $kc \gg \omega_0$ and $\nu \ll \omega_0$. We find that the solution has two branches — the electromagnetic wave with the frequency close to kc and the cyclotron wave whose frequency is

$$\omega = \omega_H \left(1 + \frac{\omega_0^2}{\omega_H^2 - k^2c^2}\right) - i\nu$$

It can be seen that the cyclotron wave slowly damps in the magnetized plasma ($\nu \ll \omega_H$). The frequency of the cyclotron wave is close to the frequency of rotation of electrons in the constant magnetic field.

Problem 3.26. Analyze the stability with respect to pinching of a plasma column produced by a linear current in a gas in an external magnetic field. Pinching involves the variation of the radius of the plasma column but does not change its axial symmetry.

Assume that the radius of the plasma column varies over a long part of the column, so that the radius of curvature of the magnetic field lines is considerably larger than the radius of the column. Then the total pressure in the column and outside it is conserved when the shape of the column is distorted. Moreover, assume that the external magnetic field H_z parallel to the current is applied to the plasma column.

Let the column radius r_0 be changed by δr_0. Since the magnetic field lines are frozen into the plasma, the magnetic flux across the plasma column $H_z \pi r_0^2$ is conserved, that is, the variation δH_z of the longitudinal magnetic field in the plasma and the variation of the column radius δr_0 are related by

$$\frac{\delta H_z}{H_z} + \frac{2\delta r_0}{r_0} = 0$$

Furthermore, the total current in the plasma is not changed by distortion of the column shape. The current is $\mathcal{J} = \frac{cr_0}{2} H_\varphi$, where H_φ is the magnetic field generated by the current. The relationship between the variation of the column radius and the variation of the magnetic field of the current is

$$\frac{\delta H_\varphi}{H_\varphi} + \frac{\delta r_0}{r_0} = 0$$

The variation of the column radius gives rise to the variation of the magnetic pressure in the plasma $\delta \frac{H_z^2}{8\pi} = \frac{H_z}{4\pi} \delta H_z$ and the variation of the magnetic pressure outside the plasma is $\frac{H_\varphi}{4\pi} \delta H_\varphi$. For instance, if the column radius decreases, the variations of the internal and externa magnetic pressures are positive and the plasma column is stable. This is the case when the internal magnetic pressure is higher than the external magnetic pressure, that is, if

$$H_z \delta H_z > H_\varphi \delta H_\varphi$$

In this case the magnetic pressure tends to restore the original shape of the column. The same condition should be satisfied if the column radius increases.

Using the relationship between the variations of the internal and external magnetic fields $\frac{\delta H_z}{H_z} = 2\frac{\delta H_\varphi}{H_\varphi}$, we find the following condition of stability of the plasma column:

$$H_z^2 > \frac{1}{2} H_\varphi^2$$

It can be seen that an external magnetic field can make a plasma column, generated by a linear current, stable with respect to pinching. If the condition formulated above is not satisfied, the plasma column may be broken. This instability develops during the time needed for a displacement of the plasma by a distance of the order of the plasma column radius and is, thus, a rapidly growing hydrodynamic instability.

Problem 3.27. Electric current flows in a weakly ionized monoatomic gas between two parallel infinite plane electrodes. The velocity distribution function for electrons has the Maxwellian form; the electron temperature is considerably higher than the gas temperature, and the frequency of elastic collisions between an electron and the gas atoms is independent of the electron velocity. Analyze the stability of current with respect to the variations of the electron number density.

Under the conditions of this problem the electric field determined by the potential difference between the electrodes is constant. Therefore, the electron drift velocity $w = eE/m\nu$ also does not change when random variations of the current density occur. Thus, the only parameters that can deviate from their equilibrium values are the number density and temperature of electrons. These parameters are related by the Saha equation or a similar equation, that is, $N_e \sim \exp(-J/T_e)$, where N_e and T_e are the equilibrium number density and temperature of electrons, and J is the ionization potential for atoms. Hence, the deviations of the electron number density N_e' and the electron temperature T_e' from the equilibrium values are related by

$$\frac{N_e'}{N_e} = \frac{J}{T_e}\frac{T_e'}{T_e}$$

and since the electron temperature is much lower than the ionization potential of the atoms ($T_e \ll J$), we have

$$\frac{N'_e}{N_e} \gg \frac{T'_e}{T_e}$$

Now let us analyze the variation of the electron energy owing to the random variation of the electron number density. If an increase in the electron number density is accompanied with an increase in the electron energy, the uniform distribution of the current over the cross section is unstable; a random increase in the electron number density in a certain region will result in its further increase. If the situation is reversed, the uniform current distribution is stable.

Let us write the balance equation for the variation of electron energy. The electron energy per unit volume is $\frac{3}{2} N_e T_e$ and the electron energy balance equation for unit volume is

$$\frac{d}{dt}\left(\frac{3}{2} N_e T_e\right) = N_e eEw - 3N_e \frac{m}{M} \nu T_e$$

The first term on the right-hand side is the energy supplied by the electric field to electrons per unit time per unit volume, and the second term is the energy transferred to the gas per unit time per unit volume. These terms have been derived in Problem 2.18 for $\nu = \text{const}$.

At equilibrium the derivative of the mean electron energy with respect to time is zero, so that $eEw = 3\frac{m}{M} \nu T_e$ (see Problem 2.19). This expression gives the electron temperature. Denote $N_e eEw$ by W_0 and write down the equation for the variation of the electron parameters for a small deviation from the equilibrium, using the above balance equation:

$$\frac{3}{2} T_e \frac{dN'_e}{dt} + \frac{3}{2} N_e \frac{dT'_e}{dt} = \frac{N'_e}{N_e} W_0 - \frac{N'_e}{N_e} W_0 - \frac{T'_e}{T_e} W_0$$

Since $T'_e/T_e \ll N'_e/N_e$, we can ignore the derivative of the electron temperature. Using the relationship between the variations of the electron temperature and the electron number density, we obtain

$$\frac{dN'_e}{dt} = -\frac{T'_e}{T_e} \frac{W_0}{\frac{3}{2} T_e} \equiv -N'_e \cdot \frac{T_e}{J} \frac{W_0}{\frac{3}{2} N_e T_e}$$

The solution of this equation shows that the uniform current distribution is stable under the conditions of this problem. Any random deviation of the electron number density from the equilibrium value decays as

$N_e' \sim \exp(-\gamma t)$, where

$$\gamma = \frac{T_e}{J} \frac{N_e eEw}{\frac{3}{2} N_e T_e} = \frac{2eEw}{3J}$$

Problem 3.28. A weakly ionized gas is between two infinite plane electrodes and a constant magnetic field is applied parallel to them. A constant potential difference is maintained between the electrodes and an electric current flows between them. Other conditions are the same as in Problem 3.27. Analyze the possibility of ionization instability due to the development of ionization in certain plasma regions.

At first, let us analyze the motion of electrons in a weakly ionized gas, since it determines the currents in the gas. Figure 3.4 illustrates the geometry of this system, which is typical of the magnetohydrodynamic generator.

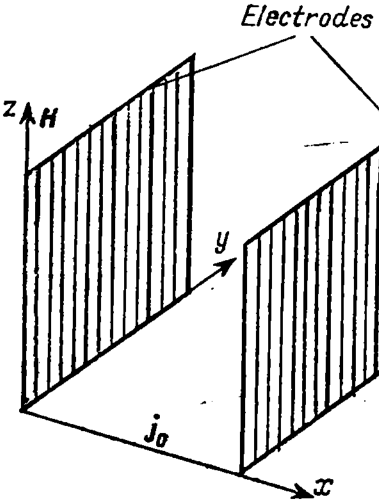

Fig. 3.4. Configuration of fields in a plasma in the case of ionization instability.

The plasma layer has a finite thickness along the x-axis parallel to which the electric current $\mathbf{j}_0 = -eN_0\mathbf{w}_0$ flows (N_0 is the equilibrium number density of electrons, and $\mathbf{w}_0$ is their mean velocity). The electric field is in the plane xy. The motion equation for electrons is

$$m\frac{d\mathbf{w}}{dt} = -e\mathbf{E} - \frac{e}{c}[\mathbf{w}\mathbf{H}] - m\mathbf{w}\nu$$

where **E** and **H** are the electric and magnetic fields, respectively, **w** is the mean velocity of electrons, and ν is the frequency of collisions between electrons and atoms, which is independent of the electron temperature under the conditions of the problem.

Writing down this equation in terms of the vector components for the steady-state case, we obtain

$$-eE_x - m\omega_H w_y - \nu m w_x = 0$$

$$-eE_y + m\omega_H w_x - \nu m w_y = 0$$

where $\omega_H = eH/mc$ is the Larmor frequency for electrons. The solution of this system of equations is

$$w_x = -\frac{e(E_x - E_y\,\omega_H/\nu)}{m\nu\,(1 + \omega_H^2/\nu^2)}, \quad w_y = -\frac{e(E_x\,\omega_H\,/\nu + E_y)}{m\nu\,(1 + \omega_H^2/\nu^2)}.$$

Since no current flows along the y-axis, the mean electron velocity along the y-axis is zero. Hence

$$E_y = -\frac{\omega_H}{\nu}E_x = \frac{m\omega_H w_x}{e}.$$

It is convenient to write down the solution in the vector form:

$$\mathbf{w} = -\frac{e\mathbf{E}\nu}{m(\omega_H^2 + \nu^2)} + \frac{e[\mathbf{E}\boldsymbol{\omega}_H]}{m(\omega_H^2 + \nu^2)}$$

Here the vector $\boldsymbol{\omega}_H = e\mathbf{H}/mc$. Since the mean electron velocity is uniquely determined by the electric field, a reasonable approach is to find the electric field from the above system of equations. We obtain

$$\mathbf{E} = -\frac{m\nu}{e}\mathbf{w} - \frac{m}{e}[\boldsymbol{\omega}_H\mathbf{w}]$$

Now let us analyze the possibility of the development of ionization instability. This can be described as follows. Let the electron number density increase in a certain plasma region. Owing to the passage of current in the plasma, evolution and absorption of heat are changed in this region. Let the energy increase be higher than the energy transferred from electrons to the gas. Then the electron number density will grow further owing to the dependence on the temperature. Thus, the process is unstable, since a random variation of the electron number density in a certain region of the plasma results in a considerable change of this parameter. This is known as the *ionization instability*.

Note that the time of the development of the ionization instability is of the order of the ionization time, that is, it is long compared with the time of motion of electrons. On the other hand, the time of the development of the ionization instability is too short for the removal of heat.

We have to find the relationship between variation of the electron energy and the electron number density during the development of the ionization instability. The heat evolved per unit volume per unit time is $\mathbf{jE}$ (here $\mathbf{j}$ is the current density and $\mathbf{E}$ is the electric field). A perturbation of the electron number density N_e' produces the perturbations $\mathbf{E}'$ and $\mathbf{w}'$ of the electric field and the mean electron velocity, respectively. Then the variation of energy transferred to electrons per unit volume per unit time is

$$W' = -eN'\mathbf{w}_0\mathbf{E}_0 - eN_0\mathbf{w}'\mathbf{E}_0 - eN_0\mathbf{w}_0\mathbf{E}'$$

where the subscript 0 denotes the unperturbed parameters.

Let us now use the relationship between the electric field and the mean electron velocity, derived above. Owing to the linearity of the respective equations, this relationship has the same form for the perturbed parameters:

$$\mathbf{E} = -\frac{m\nu}{e}\mathbf{w} - \frac{m}{e}[\boldsymbol{\omega}_H\mathbf{w}]$$

Hence we obtain

$$W' = N'mw_0^2\nu + 2N_0m\mathbf{w}_0\mathbf{w}'\nu$$

The relationship between N' and $\mathbf{w}'$ can be derived from the continuity equation for electrons, which can be used in the steady-state form $\operatorname{div}(N\mathbf{w}) = 0$, since the instability develops slowly. Writing the dependence on the coordinate $\mathbf{r}$ in the form $\exp(i\mathbf{rk})$, where $\mathbf{k}$ is the wave vector, we obtain

$$N_0(\mathbf{w}'\mathbf{k}) + N'(\mathbf{w}_0\mathbf{k}) = 0$$

Now we have to eliminate the perturbed electron velocity $\mathbf{w}'$ from the expression for W'. First, we must find the direction of $\mathbf{w}'$. Since the perturbation develops slowly, we can write $\mathbf{E}' = -\nabla\varphi'$, where φ' is the perturbation of the electric potential. As before, we assume $\varphi' \sim \exp(i\mathbf{kr})$ and obtain $\mathbf{E}' = -i\mathbf{k}\varphi'$, that is, the vectors $\mathbf{E}'$ and $\mathbf{k}$ are either parallel or antiparallel. Then, using the relationship between $\mathbf{w}'$ and $\mathbf{E}'$, we obtain

$$\mathbf{w}' = \text{const}\cdot\left(\mathbf{k} - \frac{1}{\nu}[\mathbf{k}\boldsymbol{\omega}_H]\right)$$

Multiplying the vector $\mathbf{w}'$ by itself, we find the constant correct to the sign. Finally, we have

$$\mathbf{w}' = \mp\frac{w'(\nu\mathbf{k} - [\mathbf{k}\boldsymbol{\omega}_H])}{k\sqrt{\omega_H^2 + \nu^2}}$$

Substitute this expression into the relationship derived from the continuity equation and obtain

$$w' = \mp\frac{N'}{N_0}\frac{w_0}{\nu}\sqrt{\omega_H^2 + \nu^2}\cos\alpha$$

where $\cos\alpha = k_x/k$ (α is the angle between the vectors $\mathbf{k}$ and $\mathbf{w}_0$). Substitution of this relationship into the expression for W' yields

$$W' = N'\frac{mw_0^2}{\tau}(1 - 2\cos^2\alpha + \omega_H\tau\sin 2\alpha)$$

This expression is independent of the sign chosen in the expression for $\mathbf{w}'$.

The variation of the energy transferred from electrons to the gas atoms per unit volume per unit time is determined by the elastic collisions between electrons and the gas atoms. As found in Problem 3.27, this variation, taking into account the relationship between perturbations of the temperature and the electron number density, is

$$\frac{N_e'}{N_e} W_0 \left(1 + \frac{T_e}{J}\right)$$

where $W_0 = mw_0^2 \nu N_e$ is the energy dissipated by electrons per unit volume per unit time owing to their collisions with the gas atoms.

The above expressions yield the following balance equation for the electron energy per unit volume:

$$\frac{d}{dt}\left(\frac{3}{2} N_e T_e\right) = N_e' m w_0^2 \nu \left(\frac{\omega_H}{\nu} \sin 2\alpha - \cos^2\alpha - \frac{T_e}{J}\right)$$

Let us consider the term in the parentheses on the right-hand side of the balance equation. If this term is positive, any random deviation of the electron number density from the equilibrium value continues to grow, that is, instability develops. If we regard this term as a function of the direction of the wave vector **k** describing the perturbation (or the angle α), its maximum is found at $\tan 2\alpha = -\omega_H/\nu$. At this maximum the balance equation for the electron energy density has the following form:

$$\frac{d}{dt}\left(\frac{3}{2} N_e T_e\right) = N_e' m w_0^2 \nu \left(\frac{\sqrt{\omega_H^2 + \nu^2}}{\nu} - 1 - \frac{T_e}{J}\right)$$

This shows that for the field configuration of this problem the ionization instability has a threshold, namely, it occurs when $\omega_H/\nu \geqslant \sqrt{2T_e/J}$. When the ratio ω_H/ν is high, the ionization instability develops for perturbations propagating at 45° to the current direction; when this ratio is small, the most unstable perturbations propagate almost perpendicular to the current.

Problem 3.29. Analyze the development of the current-convective instability for weakly ionized plasma of a positive gas-discharge column. The plasma is axially symmetric. The constant magnetic field $\mathbf{H}_0$ and the electric field $\mathbf{E}_0$ giving rise to electron current are applied along the axis of the plasma column. The plasma is radially nonuniform; the electrons and ions of the plasma are magnetized.

This type of instability can be described as follows. Let electrons be rotated with respect to ions in a certain cross section of the plasma column. This gives rise to the azimuthal electric field which enhances this motion. The development of the instability is slow enough for plasma quasineutrality to be conserved. Let us produce a small nonuniformity of the plasma in the direction of the current, so that the plasma density is different in the neighbouring cross sections of the column. Since the current in the plasma should be conserved, the decrease in the plasma density in a certain cross section of the column should be compensated by an additional electric field.

Now let us assume an "oblique" perturbation of the plasma density, which is compensated by an electric field directed at an angle to the column axis. Then the azimuthal component of the electric field will produce rotation of electrons and ions and will enhance a suitably directed perturbation, that is, lead to the development of instability.

Let us derive the dispersion relation for such perturbations. The plasma is quasineutral and magnetized, so that the velocity of electrons and ions is

$$\mathbf{w} = c\frac{[\mathbf{EH}_0]}{H_0^2}$$

where **E** is the electric field of the wave. The magnetic field of the wave can be ignored, since the perturbation is slow. Owing to the same reason the electric field can be described by the potential φ, that is $\mathbf{E} = -\nabla\varphi$.

The current density $j_z = \Sigma E_z$ in the plasma must be conserved (here Σ is the plasma conductivity, and E_z is the electric field component along the z-axis which is parallel to the axis of the plasma column). The electric field is the sum of the external field and the wave field. Since all the wave parameters are proportional to $\exp\,[i\,(\mathbf{kr} - \omega t)]$, we have $E_z = E_0 - ik_z\varphi$. The plasma conductivity is proportional to the electron density and we can write $\Sigma = \Sigma_0 + \Sigma' = \Sigma_0(1 + N_e'/N_0)$, where Σ_0 is the plasma conductivity and N_0 is the electron number density in the absence of perturbations, and Σ' and N_e' are the respective perturbed parameters. Using the above expressions, we can write the condition of conservation of the current density in the direction of the field in the following form:

$$-\Sigma_0 ik_z\varphi + \Sigma' E_0 = 0 \quad \text{or} \quad -ik_z\varphi + \frac{N_e'}{N_0}E_0 = 0$$

The continuity equation for electrons can be written as

$$\frac{\partial N}{\partial t} + \operatorname{div}\,(N_e\mathbf{w}) - \mathscr{D}_a\frac{\partial^2 N_e}{\partial z^2} = 0$$

where $\mathscr{D}_a$ is the ambipolar diffusion coefficient for plasma. Here we ignored the diffusion flux of electrons perpendicular to the magnetic field, assuming that this flux is small. Using the above dependence of the perturbed parameters on time and coordinates we can rewrite this equation to the first order terms as

$$(-i\omega + k_z^2\mathscr{D}_a)N_e' + w_x\frac{\partial N_0}{\partial x} = 0$$

where the x-axis has the direction of the maximum equilibrium density gradient.

The above expressions yield the azimuthal electron velocity:

$$w_x = \frac{cE_y}{H_0} = -\frac{c}{H_0} ik_y\varphi = -\frac{k_y}{k_z}\frac{cE_0}{H_0}\frac{N_e'}{N}$$

Substituting this expression into the above equation, we obtain the following dispersion relation:

$$i\omega = k_z^2\mathscr{D}_a + \frac{k_y}{k_z}\frac{cE_0}{H_0L}, \quad \text{where} \quad \frac{1}{L} = -\frac{d(\ln N_0)}{dx}$$

We see that the instability ($\mathrm{Im}\ \omega < 0$) can develop if the ratio k_y/k_z has the appropriate sign. This instability has a threshold with respect to the electric field. The condition for the magnetic field is that it should be sufficiently high as stipulated above.

The results of this problem show that the optimum conditions for the growth of the instability are found when the Larmor frequency for ions is equal to the frequency of collisions between ions and the gas particles. The threshold of the current convective instability is associated with diffusion of the charged particles. If the diffusion rate is sufficiently high, perturbations of plasma are eliminated by means of diffusion and cannot grow.

Problem 3.30. Analyze the possible development of instability in a weakly ionized gas-discharge plasma. External electric field produces current in the plasma and perturbations appearing in the plasma are assumed to travel with the current.

The external electric field maintains electron current in the plasma. If the resistance of the external circuit is much higher than the resistance of the discharge gap, the plasma current does not change with time. Perturbations of the electron number density or other plasma parameters, which can appear in the plasma, are transported with electrons. We have to analyze the stability of the plasma with respect to such perturbations.

Take the x-axis parallel to the current. The perturbations travelling with the current are assumed to be small. Using the continuity equation and the Poisson equation, we obtain

$$\frac{\partial N_e'}{\partial t} + \mathrm{div}\,\mathbf{j}_e = 0, \qquad \frac{\partial E'}{\partial x} = -4\pi e N_e'$$

where N_e' and E' are the variations of the electron number density and the electric field owing to the perturbation, and j_e is the electron flux:

$$j_e = -N_e w - \mathscr{D}\frac{\partial N_e}{\partial x}$$

The minus sign at the first term indicates that the drift velocity w of electrons has the direction opposite to the electric field, $N_e = N_0 + N_e'$ is the total electron density, and $\mathcal{D}$ is the diffusion coefficient for electrons. Assuming that the dependence of N_e' and E' on time and coordinate is described by $\exp[i(kx - \omega t)]$, we obtain the following dispersion relation for small perturbations:

$$\omega = kw - i\mathcal{D}k^2 - i4\pi eN_0 \frac{dw}{dE}$$

We have used here the relationship

$$\frac{\partial w}{\partial x} = \frac{dw}{dE}\frac{\partial E'}{\partial x} .$$

This dispersion relation shows that the velocity of propagation of the corresponding waves is equal to the drift velocity of electrons. If $dw/dE > 0$, the waves decay, and if $dw/dE < 0$, the long-wavelength oscillations grow. The so-called electric *domains* appear in the course of the development of such instability.

Let us analyze the physical nature of this instability. Figure 3.5 shows the drift velocity as a function of the electric field. Let the electron current density be $j_0 = w_0N_0$, where N_0 is the mean number density of electrons in a plasma. For simplicity we ignore diffusion of electrons; then the electron current density is $j = wN_e$. Under the given conditions the drift electron velocity w_0 can be found for the fields E_1, E_2 and E_3. If the electric field in the discharge gap is E_1 or E_3, a small increase in the electron number density produces an increase in the field and, hence, in the drift velocity, so that the electron current density increases. Since the external conditions prohibit this, such a perturbation decays.

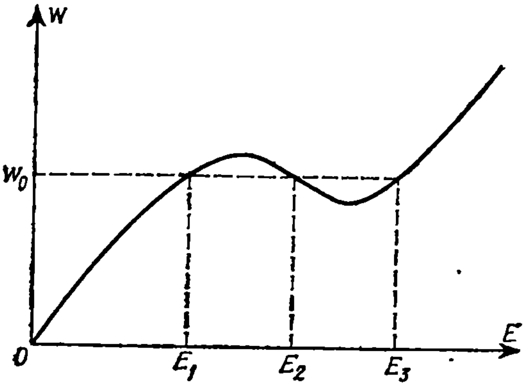

Fig. 3.5. The drift velocity of electrons as a function of electric field in the case of electric domain.

The situation is different for the electric field E_2. Then an increase in the electron number density in a given region leads to a decrease in the electric field and, hence, a decrease in the drift velocity. Thus, we can make up a solution with the perturbed electric field in the interval from E_1 to E_3 and the electron current density being equal to the specified value $w_0 N_0$. This solution is known as the electric domain.

§ 3.3. NONLINEAR PHENOMENA IN PLASMA

The development of instabilities in a plasma results in the growth of certain oscillation modes until the energy of these oscillations becomes considerable. Further development of these oscillations is determined by their interaction with the plasma. The oscillation amplitude has a significant effect on such interactions, so that the interaction processes are nonlinear. Nonlinear effects also play an important role in wave propagation, as they determine the wave shape and the character of its evolution.

Problem 3.31. Analyze decay of plasma oscillations with the frequency ω_0 and the wave vector k_0 into plasma oscillations of a lower frequency and the ion-acoustic wave (ion sound).

Let us consider the development of the ion sound and the plasma oscillations against the background of the initial plasma oscillation with the electric field

$$E = E_0 \cos (k_0 x - \omega_0 t)$$

In the zero approximation we assume that the electric field amplitude E_0 and other wave parameters are real. The motion equation for electrons $m\dfrac{dv_0}{dt} = -eE$ yields the electron velocity

$$v_0 = u_0 \sin (k_0 x - \omega_0 t), \quad \text{where} \quad u_0 = eE_0/m\omega_0$$

Let another plasma wave and the ion sound be excited in the system simultaneously with the initial plasma oscillations and let their amplitudes be small compared with the amplitude of the initial oscillations. Let us consider their development with time, taking into account their interactions with each other and with the initial oscillations. Since the velocity of ions is much lower than the velocity of electrons, we can analyze these waves separately.

Write down the motion and continuity equations for ions:

$$M\frac{dv_i}{dt} = eE$$

$$\frac{\partial}{\partial t} N_i' + N_0 \frac{\partial v_i}{\partial x} = 0$$

Here M is the ion mass, v_i is the ion velocity, N_0 is the equilibrium number density of ions, N_i' is the perturbation of the ion number density due to the oscillations, and E is the electric field due to the oscillations. Elimination of the ion velocity from these equations yields

$$\frac{\partial^2}{\partial t^2} N_i' + \frac{eN_0}{M}\frac{\partial E}{\partial x} = 0 \tag{3.32}$$

We can find the electric field from the motion equation for electrons by averaging it over fast oscillations. The one-dimensional Euler equation (1.18) can be rewritten for electrons as

$$\frac{\partial v_e}{\partial t} + v_e\frac{\partial v_e}{\partial x} + \frac{1}{mN}\frac{\partial p_e}{\partial x} + \frac{eE}{m} = 0 \tag{3.33}$$

where the electron gas pressure is $p_e = NT_e$, T_e is the electron temperature, and N is the electron number density. After averaging over fast oscillations the first term in the equation is zero. Write down the electron velocity as

$$v_e = v_0 + v_e'$$

where v_e' is the electron velocity due to the small-amplitude plasma wave. Then we have

$$\overline{v_e\frac{\partial v_e}{\partial x}} = \frac{1}{2}\overline{\frac{\partial v_e^2}{\partial x}} = \frac{1}{2}\overline{\frac{\partial}{\partial x}(v_0 + v_e')^2} = \frac{\partial}{\partial x}\overline{v_0 v_e'}$$

where the bar denotes the averaging over fast oscillations. Further, assume that $T_e = \text{const}$ (that is, the rate of energy exchange in the ion-acoustic wave is high). During the motion of ions the electron density has time to redistribute to maintain quasineutrality of the plasma. Therefore, after averaging the deviation of the electron number density from the equilibrium density is the same as that for ions, and the third term in the Euler equation becomes

$$\frac{1}{mN}\frac{\partial p_e}{\partial x} = \frac{T_e}{mN_0}\frac{\partial N_i'}{\partial x} \qquad (N \approx N_0)$$

Thus, the Euler equation (3.33) after averaging is transformed into

$$\frac{\partial}{\partial x}(\overline{v_0 v_e'}) + \frac{T_e}{mN_0}\frac{\partial N_i'}{\partial x} + \frac{eE}{m} = 0$$

Substituting the electric field derived from this equation into eq. (3.32), we obtain

$$\frac{\partial^2 N_i'}{\partial t^2} - \frac{T_e}{M}\frac{\partial^2 N_i'}{\partial x^2} - \frac{mN_0}{M}\frac{\partial^2}{\partial x^2}(\overline{v_0 v_e'}) = 0 \tag{3.34}$$

If we ignore the last term in eq. (3.34) and assume the harmonic dependence of the ion number density on time and coordinates, we obtain the dispersion relation (3.10) for the frequency ω_i and the wave vector k_i of the ion sound:

$$\omega_i = \sqrt{\frac{T_e}{M}}\, k_i$$

To take into account the interaction between the ion sound and the plasma oscillations, we have to trace the motion of electrons in the field of the small-amplitude plasma wave.

To do this, we shall use the Maxwell equation (see Problem 3.9) for the electric field of the small-amplitude plasma, assuming that its magnetic field is zero:

$$\frac{\partial E'}{\partial t} + 4\pi j' = 0$$

Here E' is the electric field of the wave, and j' is the current density generated by it.

For simplicity, we shall ignore thermal motion of electrons, since it determines only a small variation of the oscillation frequency [see eq. (3.7)]. Therefore, when writing the expression for the current density, we can ignore the variation of the electron density under the effect of the electron pressure due to the plasma wave and assume that the electron number density is $N_e + N_i'$, where N_e includes the equilibrium electron number density and its variation under the effect of the initial plasma wave, and N_i' is the variation of the number density owing to the motion of ions. Accordingly, the electron velocity is $v_0 + v_e'$, where v_0 is the electron velocity due to the initial plasma wave, and v_e' is the electron velocity due to the small-amplitude plasma wave. Hence, the current density due to the small-amplitude plasma wave is

$$j' = -e(N_e + N_i')(v_0 + v_e') + eN_e v_0 = -eN_e v_e' - eN_i' v_0$$

where the second-order terms in small amplitude are neglected. Now the Maxwell equation can be rewritten as

$$\frac{\partial E'}{\partial t} - 4\pi e N_i' v_0 - 4\pi e N_e v_e' = 0$$

The motion equation for electrons is

$$\frac{m\, dv_e'}{dt} = -eE'$$

Eliminating E' from these equations, we obtain the following equation for the electron velocity due to the small-amplitude plasma wave:

$$\frac{\partial^2 v_e'}{\partial t^2} + \omega_e^2 v_e' + \frac{N_i'}{N_e}\omega_e^2 v_0 = 0 \tag{3.35}$$

Here $\omega_e = (4\pi N_e e^2/m)^{1/2}$ is the frequency of plasma oscillations ignoring the thermal motion of electrons. We can see that if we do not take into account the interaction between the small-amplitude plasma wave and the initial plasma wave and the ion sound [that is, ignore the last term in eq. (3.35)], the frequency of the small-amplitude plasma wave is equal to the plasma frequency in accordance with the assumption used here.

Now let us solve the system of equations (3.34) and (3.35). Express the sought parameters as

$$v_0 = u_0 \sin (k_0 x - \omega_0 t), \qquad v_e = a \cos (k_e x - \omega_e t)$$

$$N_i' = bN_0 \cos (k_i x - \omega_i t)$$

where a and b are slowly varying oscillation amplitudes, ω_e and k_e are the frequency and the wave vector of the small-amplitude plasma wave, ω_i and k_i are the frequency and the wave vector of the ion sound, and N_0 is the equilibrium number density of the charged particles [in eq. (3.35) we assume $N_e = N_0$]. Since the oscillation amplitudes vary slowly and the time and coordinate dependences of the parameters are identical, we find from eqs. (3.34) and (3.35)

$$\omega_0 = \omega_e + \omega_i, \qquad k_0 = k_e + k_i \tag{3.36}$$

This condition is similar to that of parametric resonance for two coupled oscillators. Therefore, the instability we shall find is a *parametric instability*.

Taking into account the slow variation of the oscillation amplitudes and the conditions (3.36), we find from eqs. (3.34) and (3.35) the following equations for the oscillation amplitudes:

$$\frac{\partial a}{\partial t} = -\frac{\omega_e}{4} u_0 b, \qquad \frac{\partial b}{\partial t} = -\frac{mk_i}{4M\omega_i} u_0 a$$

The solution of these equations describes the oscillations' growth ($a, b \sim \exp(\gamma t)$) with the increment

$$\gamma = \frac{1}{4}\sqrt{\frac{m\omega_e}{M\omega_i}}\, u_0 k_i = \frac{1}{4}\sqrt{\frac{m\omega_e}{M\omega_i}}\,\frac{eF_0}{m\omega_0} k_i \tag{3.37}$$

Thus, the initial plasma wave is unstable and can decay into a plasma wave of a lower frequency and the ions sound. This instability is known as the *decay instability*. The growth increment for the new wave is proportional to the amplitude of the decaying wave.

Problem 3.32. When the wave amplitude is high (for instance, for plasma oscillations, ion sound, magnetosonic waves, etc.), the wave frequency can depend on the amplitude as $\omega(k) + \alpha E^2$, where $\omega(k)$ is the dependence of the wave frequency on the wave vector k for small-amplitude waves, and E is a parameter dependent on the oscillation amplitude (for instance, the electric field generated by the wave). Consider a one-dimensional packet of waves with a small spread Δk of the wave vectors ($\Delta k \ll k$). If the wave amplitude is high, nonlinear interactions can lead to the compression of the wave packet or its decay into a number of waves. This phenomenon is known as the *modulation instability*. Show that the condition of the modulation instability is

$$\alpha \frac{\partial v_g}{\partial k} < 0$$

where v_g is the group velocity of the wave.

Let us consider spreading of a one-dimensional packet of waves. The wave amplitude at the point x is

$$a(x, t) = \sum_k a(k) \exp [i(kx - \omega t)]$$

where $a(k)$ is the amplitude of the wave with the wave vector k. Let k_0 be the mean wave vector of the packet. Let us expand the wave frequency. We obtain

$$\begin{aligned}\omega &= \omega(k) + \alpha E^2 \\ &= \omega(k_0) + \left.\frac{\partial \omega}{\partial k}\right|_{k_0} (k - k_0) + \frac{1}{2} \left.\frac{\partial^2 \omega}{\partial k^2}\right|_{k_0} (k - k_0)^2 + \alpha E^2 \\ &= \omega_0 + v_g(k - k_0) + \frac{1}{2} \left.\frac{\partial v_g}{\partial k}\right|_{k_0} (k - k_0)^2 + \alpha E^2\end{aligned}$$

where $\omega_0 = \omega(k_0)$, and $v_g = \partial\omega/\partial k$ is the group velocity of the wave. Using this expansion we find

$$a(x, t) = \sum_k a(k) \exp \left[i(k - k_0)(x - x_0) - ik_0x_0 - i(k - k_0)^2 \left.\frac{\partial v_g}{\partial k}\right|_{k_0} t - i\alpha E^2(x)t \right]$$

where $x_0 = v_g t$.

This expression shows that a packet of noninteracting waves travelling with the group velocity v_g spreads at a distance of the order of its size $l \sim 1/\Delta k$ during the time of the order of $\tau \sim \left(\Delta k^2 \frac{\partial v_g}{\partial k}\right)^{-1}$. The last term in the exponent corresponding to the nonlinear interaction of waves causes the modulation of the wave packet. Compression of the wave packet or its decay into individual wave bunches can occur only if the last term and the one before the last have opposite signs. Only in this case a nonlinear interaction of waves can compensate spreading of the packet. Hence, the condition for the development of modulation instability is

$$\alpha \frac{\partial v_g}{\partial k} < 0$$

This condition is known as the *Lighthill criterion.*

Problem 3.33. Analyze the propagation of nonlinear long-wavelength oscillations of the sound type in a plasma. The dispersion relations for these waves is

$$\omega = v_g k \left(1 - \frac{r_0^2 k^2}{2}\right) \tag{3.38}$$

with the condition $kr_0 \ll 1$. Derive an equation for these waves, taking into account their dispersion and nonlinearity.

The Euler equation (1.18) for the velocity of particles in the wave has the form

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} - \frac{F}{M} = 0$$

Here $v(x, t)$ is the velocity of particles in the longitudinal wave along the x-axis, F is the force acting on the plasma particles, and M is the particle's mass. Assuming that the particle velocity in the wave is small compared with the characteristic velocities, we can write in the linear approximation $v = v_g + v'$, where v_g is the group velocity of the wave, v' is the velocity of the plasma particles in the reference frame in which the wave is at rest, and $v' \ll v_g$. Hence, in the linear approximation we obtain

$$\frac{\partial v'}{\partial t} + v_g \frac{\partial v'}{\partial x} - \frac{F}{M} = 0$$

where the last term is a linear operator with respect to v'. In the harmonic approximation we have $v' \sim \exp\,[i(kx - \omega t)]$. Let us determine the form of the operator F/M, demanding that the above equation should lead to the dispersion relation (3.38). We obtain

$$\frac{\partial v'}{\partial t} + v_g \left(\frac{\partial v'}{\partial x} + \frac{r_0^2}{2} \frac{\partial^3 v'}{\partial x^3}\right) = 0$$

The last term takes into account the weak dispersion of the long-wavelength oscillations. Now let us take into account the nonlinearity of the small-amplitude waves. The nonlinearity is determined by the second term which was linearly approximated by replacing the velocity v of the plasma particles with the wave velocity v_g. Writing now this term in its initial form to account for nonlinearity, we obtain

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{v_g r_0^2}{2}\frac{\partial^3 v}{\partial x^3} = 0 \tag{3.39}$$

This is the *Korteveg-de Vries equation* which was derived in 1895 to describe the propagation of waves in shallow waters.

Equation (3.39) takes into account simultaneously the nonlinearity and dispersion of waves and is, thus, a convenient model equation for the analysis of nonlinear dissipation processes. For slight nonlinearity conditions this equation describes the propagation of some long waves in a plasma the dispersion relation for which is given by eq. (3.38). Among such waves is the ion sound the dispersion relation (3.9) for which is identical to eq. (3.39) if the parameter r_0 in eq. (3.38) is replaced with the Debye-Hückel radius.

Problem 3.34. Analyze the propagation of a single wave in plasma using eq. (3.39).

One physically interesting class of solutions of eq. (3.39) describes so-called solitary waves or solitons. The peculiarity of solitons is that the perturbations corresponding to them do not spread out spatially with time but conserve their form. This behaviour is determined by the character of eq. (3.39). According to the dispersion relation (3.38), the shorter waves propagate with a lower velocity. Therefore, in a linear medium any perturbation is blurred owing to the difference in wave velocities. However, a slight nonlinearity of the wave can compensate its dispersion and conserve its shape.

Let us demonstrate that this is the case for the waves described by eq. (3.39). Consider a wave with the propagation velocity u, so that the dependence of the velocity of the plasma particles on time and coordinate has the form $v = f(x - ut)$. Hence, $\partial v/\partial t = -u\,\partial v/\partial x$ and eq. (3.39) is transformed into

$$(v - u)\frac{dv}{dx} + \frac{v_g r_0^2}{2}\frac{d^3 v}{dx^3} = 0$$

Assuming that perturbation is zero at a large distance from the wave, that is $v = 0$, $d^2v/dx^2 = 0$ for $x \to \infty$, we can reduce the equation to

$$\frac{v_g r_0^2}{2}\frac{d^2 v}{dx^2} = uv - \frac{v^2}{2}$$

It can be readily seen that $v = a\cosh^{-2}\alpha x$ is a solution of this equation. Substitution of this expression into the equation yields

$$v = \frac{3u}{\cosh^2 \dfrac{x}{r_0}\sqrt{\dfrac{u}{2v_g}}}$$

The wave is seen to become more narrow with increasing amplitude, so that the size of the region with perturbation is inversely proportional to the square root of the wave amplitude.

Thus, eq. (3.39) has steady-state solutions describing a nonspreading solitary wave, or soliton. Solitons can be produced in the development of a certain perturbation with time. The amplitude a and the size $1/\alpha$ of a soliton are such that the ratio a/α^2 is independent of the amplitude. If the initial amplitude of the perturbation is relatively small, this relationship will be established in the process of spreading of the wave packet and further such a perturbation will be reduced, basically, to one soliton. If the initial amplitude of the perturbation is relatively large, this perturbation will split into several solitons. Thus, solitons are not only stable steady-state perturbations in the system but also play an important part in evolution of arbitrary perturbations in a nonlinear dispersive media.

Problem 3.35. Analyze the propagation of a nonlinear ion-acoustic wave.

Equations (3.8) and the dispersion relation (3.9) for the ion sound in the linear approximation were derived above from the Euler equation, the continuity equation and the Poisson equation. Now we shall use these equations without the linear approximation:

$$\left.\begin{aligned}
&\frac{\partial v_i}{\partial t} + v_i\frac{\partial v_i}{\partial x} + \frac{e}{M}\frac{\partial \varphi}{\partial x} = 0\\
&\frac{\partial N_i}{\partial t} + \frac{\partial}{\partial x}(N_i v_i) = 0\\
&\frac{\partial^2 \varphi}{\partial x^2} = 4\pi e(N_e - N_i)
\end{aligned}\right\}$$

Here v_i is the velocity of ions in the wave, φ is the field potential, and N_e and N_i are the number densities of electrons and ions, respectively. As discussed above, owing to their high mobility electrons are at equilibrium with the field. Therefore, we can use the Boltzmann distribution $N_e = N_0 \exp(e\varphi/T_e)$, where N_0 is the mean number density of the charged plasma particles, and T_e is the electron temperature.

Let us now analyze the motion of ions in the steady-state wave, so that the plasma parameters v_i, N_i and φ depend on time and coordinate as $f(x - ut)$, where u is the velocity of the wave. Then the system of equations can be rewritten as

$$\left.\begin{aligned} & -u\frac{dv_i}{dx} + v_i\frac{dv_i}{dx} + \frac{e}{M}\frac{d\varphi}{dx} = 0 \\ & \frac{d}{dx}[N_i(v_i - u)] = 0 \\ & \frac{d^2\varphi}{dx^2} = 4\pi e[N_0 \exp(e\varphi/T_e) - N_i] \end{aligned}\right\}$$

Assuming that the perturbation is zero far from the wave, that is, $N_i = N_0$, $v_i = 0$ and $\varphi = 0$ for $x \to \infty$, we obtain from the first two equations

$$\frac{v_i^2}{2} - uv_i + \frac{e\varphi}{M} = 0, \quad N_i = N_0\frac{u}{u - v_i}$$

(Note that since $N_i > 0$, the second equation yields $v_i < u$, and, hence, the first equation yields $\varphi \geqslant 0$, that is, the electric field potential of the ion-acoustic wave is always positive.) The first equation gives $v_i = u \pm \sqrt{u^2 - 2e\varphi/M}$. Further, we find from the second equation

$$N_i = N_0\frac{N_e}{\sqrt{u^2 - \dfrac{2e\varphi}{M}}}$$

Substitution of this expression into the last equation for the ion sound yields

$$\frac{d^2\varphi}{dx^2} = 4\pi eN_0\left[\exp(e\varphi/T_e) - \frac{u}{\sqrt{u^2 - \dfrac{2e\varphi}{M}}}\right]$$

This equation describes the potential of the electric field of the nonlinear ion-acoustic wave. The equation has the form of the motion equation for a particle if φ is regarded as the coordinate and x as time. Owing to the general properties of this equation its order can be reduced by multiplying it by $d\varphi/dx$ and integrating the result. We obtain

$$\frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 - 4\pi N_0T_e\exp\left(\frac{e\varphi}{T_e}\right) - 4\pi N_0Mu\sqrt{u^2 - \frac{2e\varphi}{M}} = \text{const}$$

Assuming that at large distances from the wave the potential φ and the electric field strength $-d\varphi/dx$ of the wave are zero, we obtain

$$\frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 + 4\pi N_0T_e\left[1 - \exp\left(\frac{e\varphi}{T_e}\right)\right] + 4\pi N_0Mu\left(u - \sqrt{u^2 - \frac{2e\varphi}{M}}\right) = 0$$

This solution describes a solitary wave since the boundary conditions specify that the perturbation is absent at large distances from the wave. The solution can be used to find the shape of the soliton and the relationship between its parameters for various wave amplitudes. Let us determine the relationship between the maximum potential $\varphi_{\max}$ of the wave and its velocity u. We put $d\varphi/dx = 0$ and $\varphi = \varphi_{\max}$. Using the dimensionless variables $\zeta = e\varphi_{\max}/T_e$ and $\eta = Mu^2/2T_e$, we can rewrite the above equation as

$$1 - \exp\zeta + 2\eta(1 - \sqrt{1 - \zeta/\eta}) = 0$$

Let us consider the limiting cases. When $\zeta \to 0$, that is, for the ion-acoustic wave of a small amplitude, we have $\eta = \frac{1}{2}$, so that the phase velocity of this wave is $u = \sqrt{T_e/M}$, which is in full agreement with eq. (3.10), For large-amplitude waves we have $\zeta = \eta$ so that the equation for ζ is

$$1 - \exp\zeta + 2\zeta = 0$$

This equation yields $\zeta = 1.26$, so that $e\varphi_{\max} = 1.26T_e$ and $u = 1.58\sqrt{T_e/M}$. For higher wave amplitudes the potential at the centre of the wave becomes too high and the ions are reflected from the crest of the wave. This leads to wave separation. Thus, a solitary ion-acoustic wave can exist only in a limited range of wave amplitudes and velocities.

Problem 3.36. Derive the dispersion relation for plasma waves which have a considerable energy density. Assume that the variation of the electric field of the wave is small at distances of the order of the wavelength.

In this case oscillations of plasma affect the electron distribution and, ultimately, the character of its oscillations. Denote by

$$W(r) = E^2/8\pi$$

the energy density of the plasma oscillations [here $E(r)$ is the electric field due to oscillations]. The total pressure due to electrons is the sum of the pressure $p = N(T_e + T_i)$ due to thermal motion (N is the density of the charged particles and T_e and T_i are the temperatures of electrons and ions, respectively; below we shall assume $T_e = T_i$) and the pressure $W(r)$ due to the plasma oscillations and it is constant and uniform in plasma. The rate of establishing of the constant total pressure of electrons is equal to the electron sound speed which is of the order of the thermal velocity of electrons, or the rate of establishing of the equilibrium electron density $N(r)$ which can be found from the equation

$$NT_e + W(r) = 2N_0T_e$$

Here N_0 is the electron number density at the infinity, where plasma oscillations are assumed to be absent and the electron temperature is assumed to be spatially uniform.

Now let us analyze the dispersion relation for the plasma waves. We can derive this relation in the conventional way (see Problem 3.2) but for a limited spatial region where the variation of the electron density is small. The result is the dispersion relation (3.7)

$$\omega^2 = \omega_0^2(r) + 3\frac{T_e}{m}k^2$$

where the plasma frequency $\omega_0(r)$ varies spatially with the electron density. Using the equation for the electron density and defining ω_0^2 for the electron density at infinity, we obtain

$$\omega^2(r) = \omega_0^2\left[1 - \frac{W(r)}{2N_0T_e}\right] + 3\frac{T_e}{m}k^2$$

Since the first term is considerably larger than the other two, we can rewrite this relation as

$$\omega^2(r) = \omega_0^2\left[1 - \frac{W(r)}{2N_0T_e} + 6r_D^2k^2\right]$$

where $r_D = [T/(8\pi Ne)^2]^{\frac{1}{2}}$ is the Debye-Hückel radius for the plasma at infinity.

The dispersion relation shows that if the energy density of the plasma oscillations is high enough $[W(r) > 6r_D^2k^2]$, then the oscillations cannot exist at infinity. Thus, plasma oscillations with high energy density can become concentrated in a limited spatial region. This results in the formation of solitons. Note that, since $r_Dk \ll 1$, solitons can be formed when the energy density of oscillations is much lower than the density of thermal energy of electrons.

Note another feature of the soliton formation. According to the results of Problem 3.31, when plasma oscillations get into a region with a lower electron density, they decay into plasma oscillations with a lower frequency and the ion sound. Thus, they are confined to the spatial region with a lower electron density and the resulting solitons are stable.

Problem 3.37. Find if the Lighthill criterion is satisfied for the high-amplitude plasma oscillations.

Let us analyze the dispersion relation derived in Problem 3.36 for the Langmuir oscillations with a considerable energy density. We have

$$\omega_0^2(r) = \omega_0^2\left[1 - \frac{E^2(r)}{16\pi N_0T_e} + 6r_D^2k^2\right]$$

Substituting $E(r) = E_0 \cos \omega t$ into this relation and averaging the result over time, we obtain

$$\omega_0(r) = \omega_0 \left(1 - \frac{E_0^2}{64\pi N_0 T_e} + 3r_D^2 k^2 \right)$$

This relation can be rewritten as $\omega_0(r) = \omega(k) + \alpha E_0^2$, where $\omega(k) = \omega_0(1 + 3r_D^2 k^2)$ and $\alpha = - \omega_0/64\pi N_0 T_e$. The group velocity for the Langmuir oscillations is

$$v_g = \partial\omega/\partial k = 6T_e k/m, \quad \text{so that} \quad \alpha \partial v_g/\partial k = - 3\omega_0/32\pi N_0 m$$

and the Lighthill criterion is satisfied. Hence, the nonlinear Langmuir oscillations can give rise to a solitary wave — soliton.

Problem 3.38. Analyze the nonlinear properties of an electric domain.

An electric domain is a perturbation of the electron number density and the electric field in a plasma, which propagates with the drift velocity of electrons (see Problem 3.30). Electric domains can appear if the electric field E is such that plasma has a negative differential conductivity, that is, $dw/dE < 0$, where w is the drift velocity of electrons (see Fig. 3.5). We have to find distributions of the electric field and the electron number density in the electric domain in the direction of its propagation, assuming that the initial electric field strength corresponds to the decreasing part of the $w(E)$ plot. Let the initial field strength be E_2 in Fig. 3.5. Let us find the field distribution in the electric domain assuming that the initial perturbation was an increase in the field.

We shall make use of the Poisson equation and the constancy of the electron current density determined by the external circuit parameters. Since the electric field is antiparallel to the drift velocity of electrons, we can write

$$-\frac{\partial E}{\partial x} = 4\pi e(N_0 - N_e)$$

$$j_0 \equiv - N_0 w(E_2) = - N_e w(E) - \mathscr{D} \frac{\partial N_e}{\partial x}$$

where N_e is the electron number density, N_0 is the unperturbed electron number density, and $\mathscr{D}$ is the diffusion coefficient for electrons. Eliminating the electron number density from these equations, we obtain the equation for the electric field distribution in the domain:

$$\mathscr{D} \frac{d^2E}{dx^2} - w(E) \frac{dE}{dx} - 4\pi e N_0 [w(E) - w(E_2)] = 0$$

We shall solve this equation with the initial condition $E = E_2$ for $x = 0$.

Assume that diffusion plays a secondary part and ignore it at the first stage of the analysis. Then the equation for the electric field is transformed into

$$\frac{dE}{dx} = 4\pi e N_0 \left[\frac{w(E_2)}{w(E)} - 1\right]$$

The solution of this equation is shown by the dotted line in Fig. 3.6. We see that a small increase in E results in the term in the square brackets

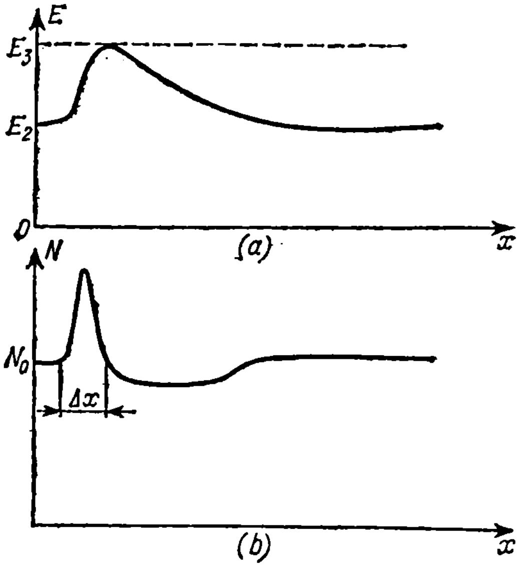

Fig. 3.6. Distributions of the electric field (*a*) and the electron number density (*b*) in the electric domain.

ecoming positive and in a further increase in E according to the equation. The electric field continues to increase up to the value E_3 for which $w(E_2) = w(E_3)$. At this point $dE/dx = 0$ and the field does not vary further. Hence, this solution corresponds to the transition of the system from the unstable state E_2 to the stable state E_3.

However, this solution does not meet other practical conditions. The growth of field strength necessitates increasing voltage across plasma; moreover, plasma quasineutrality is partially violated. These objections vanish if we take into account diffusion of electrons and make the condition that $E \to E_2$ for $x \to \infty$. The resulting solution is shown by the solid curve in Fig. 3.6. The field distribution is asymmetric since we assumed that diffusion was weak and the electric domain has a sloping front. The characteristic size of the front of an electric domain is of the order of $\mathcal{D}/w$. The characteristic size of the trailing edge of an electric domain under the assumption of weak diffusion can be found from the above equation neglecting diffusion and is given by

$$\Delta x = \int_{E_2}^{E_3} \frac{dE}{4\pi e N_0} \left[\frac{w(E_2)}{w(E)} - 1\right]^{-1}$$

CHAPTER 4

PROCESSES INVOLVING CHARGED PARTICLES IN GASES

This chapter discusses processes involving charged particles in gases, such as birth and disappearance of charged particles in a weakly ionized plasma or their motion in external fields. We shall describe these processes (rather than their effect on the properties and kinetics of a weakly ionized plasma). The main attention will be paid to the calculation of the parameters of such processes.

§ 4.1. MOTION OF IONS IN A GAS IN EXTERNAL FIELDS

According to eq. (1.76), the flux of charged particles in a gas in external fields can be written as

$$\mathbf{j} = \mathbf{w}N - \mathscr{D}\nabla N$$

where N is the number density of the charged particles, $\mathbf{w}$ is their translational velocity, and $\mathscr{D}$ is the diffusion coefficient. It is assumed here that the mean free path of the particles is small compared with the characteristic dimensions of the system. If the translational motion of the charged particles is due to a uniform electric field, a convenient parameter is the mobility

$$K = w/E$$

of the charged particles in a gas, where E is the electric field strength. For low-strength fields the mobility of the charged particles is independent of the field.

Problem 4.1. Derive the expression for the mobility of ions in a gas in the Chapman-Enskog approximation for low-strength electric fields.

In weak electric fields the energy acquired by an ion from the field in the time interval between two successive collisions is of the order of

$eE\lambda \sim eE/N\sigma$ (where $\lambda \sim 1/N\sigma$ is the mean free path) and much lower than the mean thermal energy T, that is,

$$\frac{eE}{TN\sigma} \ll 1 \tag{4.1}$$

Under this condition the drift velocity of an ion is much lower than its thermal velocity. Therefore, the velocity distribution function for ions differs little from the Maxwellian distribution $\varphi(v)$ and can be written as

$$f(\mathbf{v}) = \varphi(v)\,[1 + v\cos\theta\,\psi\,(v)] \tag{4.2}$$

where θ is the angle between the vectors $\mathbf{v}$ and $\mathbf{E}$.

Adding eq. (1.22) to eq. (4.2), we can approximately derive the mobility. If we assume that $\psi(v)$ from eq. (4.2) is independent of the collision velocity and find this function from eq. (1.22), we obtain the following expression for the mobility K_I in the first Chapman-Enskog approximation:

$$K_\mathrm{I} = \frac{3\sqrt{\pi}e}{8N\bar{\sigma}(2T_\mu)^{1/2}}, \quad w_\mathrm{I} = K_\mathrm{I}E \tag{4.3}$$

where

$$\bar{\sigma} = \frac{1}{2}\int_0^\infty \sigma^*(x)e^{-x}x^2\,dx, \quad x = \frac{\mu g^2}{2T} \tag{4.4}$$

Typically, the mobility is determined for the number density of the gas particles $N = 2.69 \times 10^{19}$ cm^{-3}, corresponding to the gas temperature of 0°C and the pressure of 1 atm. If in eq. (4.3) we take the gas temperature in kelvins, the reduced mass μ in the proton mass units, and the mean cross section $\bar{\sigma}$ in the πa_0^2 units ($= 10.88 \times 10^{-16}$ cm^{-2}), this equation is transformed into

$$K_\mathrm{I} = \frac{2.1 \times 10^4}{\bar{\sigma}(\mu T)^{1/2}}\,\mathrm{cm\cdot s^{-1}/V\cdot cm^{-1}} \tag{4.5}$$

Problem 4.2. Find the drift velocity of ions in a gas if the frequency of collisions between ions and the gas particles is independent of the collision velocity.

The drift velocity is the mean velocity of ions in a gas. It can be found from the integral relation (1.22) which equates the variations of the ion momentum per unit time owing to the interaction with the electric field and the collisions with gas particles:

$$eE = \mu N\langle g_x g\sigma^*(g)\rangle \tag{4.6}$$

Here E is the electric field, μ is the reduced mass of the ion and the gas particle, N is the number density of the gas particles, g is the relative velocity of collision, σ^* is the diffusion cross section of scattering of the ion by the gas particle, and the electric field is parallel to the x-axis.

Let us define the frequency of collisions between ions and gas particles $\nu = Ng\sigma^*(g)$. Since the frequency of collisions is independent of the collision velocity and the mean velocity of the gas particles is zero, we obtain the following expression for the mean velocity of ions:

$$w = eE/\mu\nu \tag{4.7}$$

Note that eq. (4.7) is valid for any electric field strength.

| **Problem 4.3.** Calculate the mobility of ions in a host gas.

If foreign ions travel in a host gas, then scattering of the ions by the gas molecules or atoms is determined mainly by the polarization interaction between the ion and the atom with the potential $U(R) = -\beta e^2/2R^4$, where β is the polarizability of the atom, and R is the distance between the nuclei of the atom and the ion. The process of scattering here can be described as follows. If the impact parameter is small, an ion is captured by an atom, that is, the distance between them becomes of the order of their size when repulsion occurs. Assuming that for such separation distances scattering is isotropic and ignoring scattering with large impact parameters, when the ion is not captured by the atom, we come to the conclusion that the diffusion cross section of scattering in this case is equal to the cross section of capture of the ion by the atom.

Let us find the ion-capture cross section. First, we shall determine the relationship between the impact parameter ρ and the distance $r_{\min}$ of closest approach for the ion and the atom. In the centre-of-mass frame of reference the normal component of the velocity at the distance of closest approach is zero, and the energy conservation law yields

$$\frac{\mu g_\tau^2}{2} + U(r_{\min}) = \frac{\mu g^2}{2}$$

where μ is the reduced mass of the ion and the atom, g_τ is the relative velocity of the ion and the atom at the distance of closest approach, g is the relative velocity of the ion and the atom at an infinite distance between them, and $U(r_{\min}) = -\beta e^2/2r_{\min}^4$ is the potential of interaction between the ion and the atom at the distance of closest approach. Conservation of momentum in the centre-of-mass reference frame yields $\mu g_\tau r_{\min} = \mu g \rho$. Eliminating g_τ from the above two equations, we obtain

$$\rho^2 = r_{\min}^2 + \frac{\beta e^2}{\mu g^2 r_{\min}^2}$$

Figure 4.1 shows the impact parameter as a function of the distance of closest approach for the given interaction potential. It can be seen that in this case impact parameters of collisions can exceed a certain value ρ_c given by the expression

$$\rho_c^2 = 2\left(\frac{\beta e^2}{\mu g^2}\right)^{1/2}$$

For smaller impact parameters and the given interaction potential the distance between the particles decreases to $r_{\min} = 0$, that is, one particle falls on the other and the cross section of capture of the ion by the atom is

$$\sigma_c = \pi\rho_c^2 = 2\pi\left(\frac{\beta e^2}{\mu g^2}\right)^{1/2}$$

Since the diffusion cross section of scattering of the ion by the atom coincides with the cross section of capture of the ion by the atom under the above assumptions, the frequency of the ion-atom collisions is

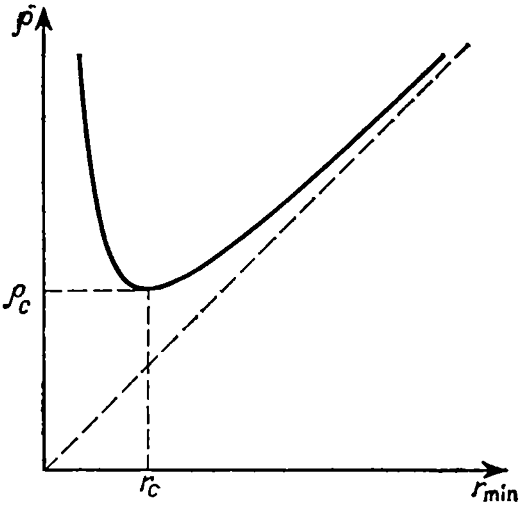

Fig. 4.1. The impact parameter as a function of the distance of closest approach of the ion and the atom. When $r_{\min}<r_c$, the ion and the atom are in the bound state; when $\rho < \rho_c$, the ion is captured by the atom and the distance between them is greatly decreased.

$$\nu = 2\pi N\left(\frac{\beta e^2}{\mu}\right)^{1/2}$$

The collision frequency is independent of the collision velocity, and eq. (4.7) gives the following expression for the ion drift velocity:

$$w = \frac{E}{2\pi N(\beta\mu)^{1/2}} \tag{4.8}$$

This expression is sufficiently close to the exact formula for the polarization interaction between the ion and the atom but is not identical to it since we approximated the nature of scattering. The drift velocity given by the exact formula is lower by about 10% than the value given by eq. (4.8).

Problem 4.4. Find the mobility of atomic ions in the parent gas in the first Chapman-Enskog approximation.

The mobility of ions in this case is determined by the ion-atom resonant charge transfer. The cross section for resonant charge transfer between an ion and the respective atom is considerably greater than the cross section of the elastic scattering of these particles even at thermal collision velocities.

Therefore, in this problem we can ignore elastic scattering of an ion by an atom; Fig. 4.2 illustrates charge transfer which can be described as a relay process (the *Sena effect*). We see in the figure that the initial atom is transformed into an ion, so that the scattering of the ion occurs even though there is no elastic scattering of particles.

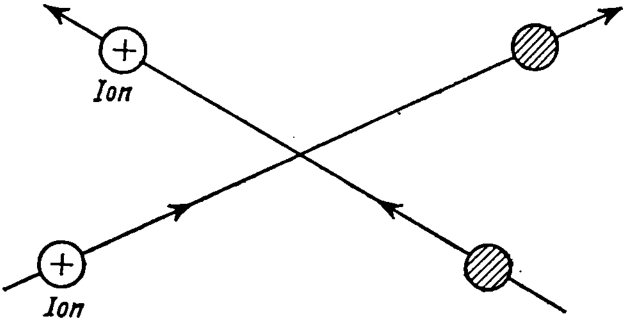

Fig. 4.2. Scattering of ion in the resonance charge transfer. The trajectories of particle are the straight lines but after the charge transfer the atom is transformed into an ion and this determines scattering of the ion

Let us find the diffusion cross section of scattering of an ion by an atom if resonant charge transfer occurs simultaneously with the elastic scattering. Denote by $W(\rho)$ the probability of resonant charge transfer for the impact parameter ρ. According to the definition of the diffusion cross section of scattering, we have

$$\sigma^* = \int_0^\infty [1 - W(\rho)]\,(1 - \cos\theta)\,2\pi\rho\,d\rho + \int_0^\infty W(\rho)\,(1 - \cos\theta')\,2\pi\rho\,d\rho$$

where θ is the scattering angle for the nucleus which initially belonged to the ion in the centre-of-mass reference frame, θ' is the angle between the velocity of this nucleus before the collision and the velocity of another nucleus after collision in the centre-of-mass reference frame; this angle describes the scattering of the ion in the case of resonant charge transfer. We have $\theta' + \theta = \pi$, so that

$$\sigma^* = \int_0^\infty 2\pi\rho\,d\rho(1 - \cos\theta) + 2\int_0^\infty \cos\theta\,W(\rho)\cdot 2\pi\rho\,d\rho$$

Hence we see that if there is no elastic scattering of an ion by an atom ($\theta = 0$), the diffusion cross section of scattering of the ion by the atom is

$$\sigma^* = 2\int_0^\infty W(\rho)\,2\pi\rho\,d\rho = 2\sigma_{\mathrm{res}} \qquad (4.9)$$

where σ_{res} is the cross section of resonant charge transfer.

Substitute this expression into eq. (4.3) for the mobility and take into account the weak dependence of the resonant charge transfer cross section on the collision velocity. The mobility of ions in the parent gas in the first Chapman-Enskog approximation is

$$K_I = \frac{3\sqrt{\pi}e}{16\sqrt{TM}N\sigma_{\mathrm{res}}} \tag{4.10}$$

where $M = 2\mu$ is the mass of the ion or the atom. Since the velocity dependence of the resonant charge transfer cross section is weak, we shall substitute into eq. (4.10) the value of σ_{res} for such a collision velocity for which the integrand in eq. (4.4) for the mean diffusion cross section is maximum. This collision velocity is ($\mu g^2/2T = 2.5$)

$$g = \sqrt{5T/\mu} = \sqrt{10T/M}$$

Problem 4.5. Calculate the mobility of ions in the parent gas for strong electric fields.

In this case the electric field satisfies a condition which is the reverse of the condition (4.1) and the drift velocity of ions is much higher than the thermal velocity of atoms. Therefore, atoms can be assumed to be stationary and ions stop moving after each charge transfer event and then are accelerated by the electric field. The probability $W(t)$ that no collision between the ion and an atom occurs in the time t after a charge transfer event can be found from the equation

$$\frac{dW}{dt} = -\nu W, \qquad W = \exp\left(-\int_0^t \nu\,dt'\right) \tag{4.11}$$

Here $\nu = Nv_x\sigma_{\mathrm{res}}$, σ_{res} is the resonant charge transfer cross section, and v_x is the relative velocity of collision which coincides with the component of the ion velocity parallel to the field, since other velocity components are small. The ion velocity v_x at the moment t after a collision is found from the motion equation for the ion $M\dfrac{dv_x}{dt} = eE$, which gives $v_x = \dfrac{eE}{M}t$. Expressing time in eq. (4.11) in terms of velocity, we find that the relative number of ions with the velocity v_x is

$$f(v_x) = C\exp\left[-\int_0^{v_x} Nv_x'\sigma_{\mathrm{res}}(v_x')\frac{M}{Ee}\,dv_x'\right], \qquad v_x > 0 \tag{4.12}$$

This is the velocity distribution function for ions; the constant C can be found from the normalization condition for the distribution function.

If we assume that the resonant charge transfer cross section is independent of the collision velocity, eq. (4.12) yields the following expression for the ion mobility in strong electric fields:

$$K = \left(\frac{2e}{\pi M N E \sigma_{\text{res}}} \right)^{1/2}$$

Problem 4.6. Find the energy distribution function for the atomic ions in the parent gas in crossed uniform electric and magnetic fields. The mean velocity of ions is much higher than the thermal velocity of atoms and the Larmor frequency for ions is much higher than the characteristic frequency of charge transfer. Calculate the mean drift velocity of ions.

As in Problem 4.5, resonant charge transfer from ions to atoms determines the process of decelerating of ions. After each charge transfer an atom is transformed into an ion, so that the ion stops after the charge transfer event (it is assumed that the mean velocity of ions is much higher than the thermal velocity of atoms). Then the ion is accelerated by the external fields until the next charge transfer event stops it. The probability W that no charge transfer occurs after the time t following the previous charge transfer is given by the equation $dW/dt = -\nu W$, which yields $W = \exp\left(-\int_0^t \nu \, dt' \right)$. Here $\nu = v/\lambda$ is the frequency of charge transfer, v is the ion velocity, $\lambda = 1/N\sigma_{\text{res}}$ is the mean free path of the ion, N is the number density of atoms, σ_{res} is the resonant charge transfer cross section, and the ion mean free path is independent of the collision velocity. Since the time t determines the ion velocity, the velocity distribution function for ions is

$$f(v)\, dv = \exp\left(-\int_0^t \nu \, dt' \right) \nu \, dt$$

Let us find the ion velocity at the moment t after a charge transfer event. Take the electric field $\mathbf{E}$ parallel to the x-axis and the magnetic field $\mathbf{H}$ parallel to the z-axis. The motion equation for the ion is

$$\frac{d\mathbf{v}}{dt} = \mathbf{a} + [\mathbf{v}\boldsymbol{\omega}_0]$$

where

$$\mathbf{a} = e\mathbf{E}/M, \qquad \boldsymbol{\omega}_0 = e\mathbf{H}/Mc$$

and M is the ionic mass. This equation can be rewritten in the vector components as

$$\frac{dv_x}{dt} = a + v_y\omega_0, \qquad \frac{dv_y}{dt} = -v_x\omega_0, \qquad \frac{dv_z}{dt} = 0$$

The solution of this system of equations for the initial conditions $v = 0$ for $t = 0$ is

$$v_x = \frac{a}{\omega_0} \sin \omega_0 t, \qquad v_y = -\frac{a}{\omega_0}(1 - \cos \omega_0 t), \qquad v_z = 0,$$

$$v = \sqrt{v_x^2 + v_y^2} = \frac{2a}{\omega_0}\left|\sin \frac{\omega_0 t}{2}\right|$$

Since the Larmor frequency is high, we obtain the following expression for the ion distribution functions:

$$f(v)\, dv = \exp(-\nu_0 t)\, \nu_0\, dt$$

Here $\nu_0 = 4a/\pi\lambda\omega_0$. The drift velocity of ions found from this expression assuming that $\nu_0 \ll \omega_0$ is

$$\overline{v}_x = a\nu_0/\omega_0^2, \qquad \overline{v}_y = -a/\omega_0$$

The drift velocity of ions is high compared with the thermal velocity of atoms, that is,

$$a/\omega_0 \gg \sqrt{T/M}$$

Problem 4.7. Find the ion distribution over the velocity components perpendicular to the electric field. Scattering of ions by the gas atoms is due to the resonant charge transfer and elastic scattering can be ignored.

When the ion with the velocity $\mathbf{v}$ and the atom with the velocity $\mathbf{v}'$ collide and resonant charge transfer occurs, the velocity of the ion will change to $\mathbf{v}'$ and the velocity of the atom to $\mathbf{v}$. Take the electric field parallel to the x-axis. Assume that the velocity distribution function can be expressed as the product of the distribution over the longitudinal velocity components $f(v_x)$ and the distribution over the transverse velocity components $F(v_\perp)$. Here v_x is the ion velocity component in the direction of the electric field and $v_\perp$ is the ion velocity component perpendicular to the field. Taking into account the character of scattering, we can write the following balance equation for the distribution $F(v_\perp)$:

$$F(v_\perp)\int \sigma_{\text{res}}(|\mathbf{v} - \mathbf{v}'|)\, |\mathbf{v} - \mathbf{v}'|\, f(v_x)\, \varphi(v_x')\, \varphi(v_\perp')\, dv_x\, d\mathbf{v}'$$

$$= \varphi(v_\perp)\int \sigma_{\text{res}}(|\mathbf{v} - \mathbf{v}'|)\, |\mathbf{v} - \mathbf{v}'| f(v_x')\, F(v_\perp')\varphi(v_x)\, dv_x\, d\mathbf{v}'$$

Here σ_{res} is the resonant charge transfer cross section which depends only on the relative collision velocity, and $\varphi(v_x)$ and $\varphi(v_\perp)$ are the Maxwellian distributions of ions or atoms over the velocity components parallel or perpendicular to the electric field. The equation means that the number of ions with a given transverse velocity component which disappear owing to the charge transfer (the left-hand side) is equal to the number of ions with the same transverse velocity component produced in this process (the right-hand side).

It can be readily seen that the solution of this equation is the Maxwellian velocity distribution

$$F(v_\perp) = \varphi(v_\perp) \equiv \frac{M}{T} \exp\left(-\frac{Mv_\perp^2}{2T}\right)$$

where T is the gas temperature, and M is the mass of the ion or the atom. Let us discuss the physical meaning of this solution.

The only assumption we used here is that the ion velocity distribution function can be expressed as the product of the distribution over the longitudinal velocity component and the distribution over the transverse velocity component. The assumption means that the transverse motion of ions is independent of their longitudinal motion, since the transverse velocity of the ion produced by charge transfer is independent of its longitudinal velocity. Since the external electric field affects only the longitudinal motion of ions, the transverse motion of ions under the conditions of the problem is independent of the field strength and the distribution of ions over the transverse velocity components is identical to the respective distribution of the atoms from which the ions have been produced. Therefore, the distribution of ions over the transverse velocity component has the Maxwellian form and it is true for any electric field strength.

Problem 4.8. Calculate the mobility of ions in the parent gas in the second Chapman-Enskog approximation for low electric field strengths.

In this case the scattering of an ion by atoms results in the exchange of velocities between them. Therefore, the kinetic equation for the velocity distribution of ions is relatively simple, so that the Chapman-Enskog expansion is not too cumbersome. We shall find here the second-order terms of this expansion.

The kinetic equation for the velocity distribution function is

$$\frac{eE}{M}\frac{\partial f}{\partial v_x} = \int [f(\mathbf{v}')\varphi(v) - f(\mathbf{v})\varphi(v')] \, |\mathbf{v} - \mathbf{v}'| \, \sigma_{\text{res}} N \, d'\mathbf{v} \tag{4.13}$$

Here the x-axis is parallel to the external electric field $\mathbf{E}$, $f(\mathbf{v})$ is the ion velocity distribution function, $\varphi(v)$ is the Maxwellian distribution for atoms (both distributions are normalized to unity), N is the number density of atoms, and σ_{res} is the resonant charge transfer cross section which is assumed to be independent of the relative collision velocity.

Expand the ion velocity distribution function in powers of the small parameter $eE\lambda/T$, where $\lambda = (N\sigma_{res})^{-1}$ is the mean free path of ions in the atomic gas and T is the gas temperature:

$$f(\mathbf{v}) = \varphi(v)\left[1 + \frac{eE\lambda}{T} v_x \psi(v)\right]$$

Using the first approximation of the above expansion, we can rewrite the kinetic equation as

$$v_x \varphi(v) = \int |\mathbf{v} - \mathbf{v}'|\, \varphi(v)\, \varphi(v')[v_x\psi(v) - v'_x\psi(v')]\, d\mathbf{v}' \qquad (4.14)$$

taking into account that $\partial\varphi(v)/\partial v_x = -Mv_x\varphi(v)/T$. Now we have to find the function $\psi(v)$. According to the Chapman-Enskog approximation, write down this function as the expansion in the Sonin polynomials:

$$\psi(v) = \sum_{n=0}^{\infty} a_n S^n_{3/2}\left(\frac{mv^2}{2T}\right)$$

Here $S^0_{3/2} = 1$ and $S^1_{3/2} = 5/2 - x$. Using this expansion and the above expression for the velocity distribution, we can find the ion mobility:

$$K = \frac{\langle v_x \rangle}{E} = \frac{1}{E}\int v_x\, d\mathbf{v}\, \frac{eE\lambda}{T} v_x \psi(v)$$

$$= \frac{2e\lambda}{3M}\int_0^\infty x \cdot \frac{2}{\sqrt{\pi}} x^{1/2} \exp(-x)\, dx \sum_{n=0}^{\infty} a_n S^n_{3/2}(x) = \frac{e\lambda}{M} a_0$$

where $x = Mv^2/2T$.

Thus, the ion mobility is expressed in terms of the coefficient at the first term of the expansion of the function $\psi(v)$ in the Sonin polynomials.

Using the general Chapman-Enskog approximation, we multiply eq. (4.14) by $\dfrac{Mv_x}{T} S^m_{3/2}(x)$ and integrate the result over the ion velocities. Then we obtain a system of equations for the coefficients a_n, which we approximate to the required order, and solve. Thus, we can find the sought parameters to the required order of approximation.

Let us apply this procedure to the left-hand side of eq. (4.14). We have

$$\int v_x \varphi\, d\mathbf{v}\, \frac{Mv_x}{T} S^m_{3/2}(x) = 2\int_0^\infty \frac{x}{3}\frac{2}{\sqrt{\pi}} x^{1/2} \exp(-x)\, dx\, S^m_{3/2}(x) = \delta_{m0}$$

where we have used the Maxwellian form of the distribution function $\varphi(v)$, $x = Mv^2/2T$, and $\delta_{m0} = 1$ for $m = 0$ and $\delta_{m0} = 0$ for $m \neq 0$. Next, we apply the same procedure to the right-hand side of eq. (4.14) and obtain the following system of equations for the coefficients a_n

$$\delta_{m0} = \int \frac{Mv_x}{T} S^m_{3/2}\left(\frac{Mv^2}{2T}\right)|\mathbf{v} - \mathbf{v}'|\, \varphi(v)\varphi(v')$$

$$\times \sum_n a_n \left[v_x S^n_{3/2}\left(\frac{Mv^2}{2T}\right) - v'_x S^n_{3/2}\left(\frac{Mv'^2}{2T}\right)\right] d\mathbf{v}\, d\mathbf{v}'$$

Symmetrization of the integrand transforms this system of equations into

$$\sum_n a_n \frac{M}{2T} \int \left[v_x S^m_{3/2}\left(\frac{Mv^2}{2T}\right) - v'_x S^m_{3/2}\left(\frac{Mv'^2}{2T}\right)\right]$$

$$\times \left[v_x S^n_{3/2}\left(\frac{Mv^2}{2T}\right) - v'_x S^n_{3/2}\left(\frac{Mv'^2}{2T}\right)\right] |\mathbf{v} - \mathbf{v}'|\varphi(v)\varphi(v') dv dv' = \delta_{m0}$$

(note that the coefficients a_n have the dimension of $\sqrt{M/T}$).

Now we solve this system of equations in the second Chapman-Enskog approximation, that is, take only two terms of the expansion of the distribution function in the Sonin polynomials and two equations from the system. Introducing the dimensionless velocities of the ion $\mathbf{u} = \sqrt{\frac{M}{2T}}\,\mathbf{v}$ and the atom $\mathbf{u}' = \sqrt{\frac{M}{2T}}\,\mathbf{v}'$, we obtain in this approximation

$$a_0\langle (u_x - u'_x)^2\, |\mathbf{u} - \mathbf{u}'| \rangle + a_1 \left\langle (u_x - u'_x)\left[u_x\left(\frac{5}{2} - u^2\right) - u'_x\left(\frac{5}{2} - u'^2\right)\right] |\mathbf{u} - \mathbf{u}'| \right\rangle = \sqrt{\frac{M}{2T}}$$

$$a_0 \left\langle \left[u_x\left(\frac{5}{2} - u^2\right) - u'_x\left(\frac{5}{2} - u'^2\right)\right](u_x - u'_x)|\mathbf{u} - \mathbf{u}'| \right\rangle + a_1 \left\langle \left[u_x\left(\frac{5}{2} - u^2\right) - u'_x\left(\frac{5}{2} - u'^2\right)\right]^2 |\mathbf{u} - \mathbf{u}'| \right\rangle = 0$$

Here the angle brackets denote averaging over the Maxwellian velocity distribution.

Introducing the dimensionless relative velocity $\mathbf{s} = \mathbf{u} - \mathbf{u}'$ and the dimensionless centre-of-mass velocity $\mathbf{G} = \frac{1}{2}(\mathbf{u} + \mathbf{u}')$, we obtain

$$u_x\left(\frac{5}{2} - u^2\right) - u'_x\left(\frac{5}{2} - u'^2\right) = s_x\left(\frac{5}{2} - G^2 - \frac{1}{4}s^2\right) - 2G_x\mathbf{G}\mathbf{s},$$

and the system of equations is reduced to

$$a_0\langle s_x^2 s\rangle + a_1\left\langle s_x s\left[s_x\left(\frac{5}{2} - G^2 - \frac{1}{4}s^2\right) - 2G_x\mathbf{Gs}\right]\right\rangle = \sqrt{\frac{M}{2T}}$$

$$a_0\left\langle s_x s\left[s_x\left(\frac{5}{2} - G^2 - \frac{1}{4}s^2\right) - 2G_x\mathbf{Gs}\right]\right\rangle$$
$$+ a_1\left\langle s\left[s_x\left(\frac{5}{2} - G^2 - \frac{s^2}{4}\right) - 2G_x\mathbf{Gs}\right]^2\right\rangle = 0$$

For instance, in the first Chapman-Enskog approximation (that is, taking $a_1 = 0$ in the first equation) we obtain $a_0\frac{\langle s^3\rangle}{3} = \sqrt{\frac{M}{2T}}$ and $a_0 = = \frac{3}{\langle s^3\rangle}\sqrt{\frac{M}{2T}} = \frac{3\sqrt{\pi M}}{16\sqrt{T}}$. Hence, the mobility is given by

$$K = \frac{e\lambda a_0}{M} = \frac{3\sqrt{\pi}\,\lambda}{16\sqrt{TM}}$$

In this approximation this result corresponds to eq. (4.3).

By averaging over the angles, we transform the system of equations into

$$a_0\frac{\langle s^3\rangle}{3} + a_1\left(\frac{5}{6}\langle s^3\rangle - \frac{5}{2}\langle s^3\rangle\langle G^2\rangle - \frac{\langle s^5\rangle}{12)}\right) = \sqrt{\frac{M}{2T}}$$

$$a_0\left(\frac{5}{6}\langle s^3\rangle - \frac{5}{9}\langle s^3\rangle\langle G^2\rangle - \frac{\langle s^5\rangle}{12}\right)$$
$$+ a_1\left[\langle s^3\rangle\left(\frac{25}{12} - \frac{25}{3}\langle G^2\rangle + \frac{}{3}\langle G^4\rangle\right)\right.$$
$$\left. + \langle s^5\rangle\left(\frac{5}{18}\langle G^2\rangle - \frac{5}{12}\right) + \frac{\langle s^7\rangle}{48}\right] = 0$$

Calculating the mean velocities, we can rewrite this system of equations as

$$\frac{a_0}{3}\langle s^3\rangle - \frac{a_1}{12}\langle s^3\rangle = \sqrt{\frac{M}{2T}}$$

$$-\frac{a_0}{12}\langle s^3\rangle - \frac{31}{24}a_1\langle s^3\rangle = 0$$

The solution of this system of equations is

$$a_0^{(2)} = \frac{62}{63}\, a_0^{(1)} = \frac{31\sqrt{\pi M}}{168\sqrt{T}}$$

where the superscript denotes the order of approximation. The difference between the results obtained in the first and the second Chapman-Enskog approximation is seen to be less than 2%. This demonstrates the good convergence of the Chapman-Enskog expansion. As follows from the above results, the ion mobility in the second Chapman-Enskog approximation is

$$K_{\text{II}}{}' = \frac{e\lambda}{M}\, a_0^{(2)} = \frac{31\sqrt{\pi}}{168\sqrt{TM}\, N\sigma_{\text{res}}} = \frac{0.327}{\sqrt{TM}\, N\sigma_{\text{res}}}$$

Problem 4.9. Find the correction for the field in the mobility of ions in the parent gas in a weak uniform electric field.

Expand the velocity distribution function for ions in powers of a small parameter proportional to the electric field strength. Since the small parameter is a scalar, it should include the product **Ev**, where **E** is the electric field and **v** is the ion velocity. We can write

$$f(\mathbf{v}) = \varphi(v)\left[1 + \sum_{n=1}^{\infty} \xi^n v_x^n \chi_n(v)\right]$$

where $\varphi(v)$ is the Maxwellian velocity distribution for ions, $\xi = eE\lambda/T$ is the dimensionless parameter, λ is the mean free path of ions, and v_x is the ion velocity component parallel to the electric field.

Substituting this expansion into the kinetic equations and taking the terms of the same order in the field strength, we obtain a system of equations relating the functions χ_n to the functions with the preceding and subsequent subscript values. Solving this system of equations, we can find the function χ_n. Then the ion mobility is

$$K = \frac{\langle v_x \rangle}{E} \frac{\sum_{r=1}^{\infty} \langle \xi^n v_x^{n+1} \chi_n(v) \rangle}{E} = \frac{e\lambda}{T}\left[\langle v_x^2 \chi_1 \rangle + \xi^2 \langle v_x^4 \chi_3 \rangle + \ldots\right]$$

where the expansion is in powers of a small parameter proportional to E^2 and the angle brackets denote the averaging over Maxwellian velocity distributions for the ion and the atom.

Let us find the first correction term for the ion mobility in this case. Substituting the expansion for the ion velocity distribution into eq. (4.13) and equating the terms with E^n, we obtain

$$\frac{T}{M}\frac{\partial}{\partial v_x}[v_x^{n-1}\chi_{n-1}(v)\,\varphi(v)] = \int|\mathbf{v}-\mathbf{v}'|\varphi(v)\varphi(v')[v_x'^n\chi_n(v') - v_x^n\chi_n(v)]\,d\mathbf{v}'$$

According to the results of Problem 4.8, we can assume that the functions χ_n are independent of the ion velocity to the expansion terms of a not too high order. Then the system of equations for χ_n can be written as

$$\chi_{n-1}\frac{T}{M}\frac{\partial}{\partial v_x}[v_x^{n-1}\varphi(v)] = \chi_n\int|\mathbf{v}-\mathbf{v}'|\varphi(v)\varphi(v')(v_n'^n - v_x^n)\,d\mathbf{v}'$$

Multiply the equations by v_x^n and integrate the results over the ion velocities. Integration of the left-hand side of the equation by parts and symmetrization of the right-hand side yield

$$\chi_{n-1}\frac{T}{M}n\langle v_x^{2n-2}\rangle = \frac{1}{2}\chi_n\langle|\mathbf{v}-\mathbf{v}'|(v_x^n - v_x'^n)^2\rangle.$$

In this way we have come to a recurrence relation which makes it possible to find the terms of the expansion of the distribution function in powers of the field from the preceding terms of this expansions. For instance, we obtain

$$\chi_2 = \frac{4T}{M}\chi_1\frac{\langle v_x^2\rangle}{\langle|\mathbf{v}-\mathbf{v}'|(v_x^2 - v_x'^2)\rangle}$$

$$\chi_3 = \frac{6T}{M}\chi_2\frac{\langle v_x^4\rangle}{\langle|\mathbf{v}-\mathbf{v}'|(v_x^3 - v_x'^3)^2\rangle}.$$

Hence, we can find the first two terms of the expansion of the ion mobility in powers of the field:

$$K = \frac{e\lambda}{M}(\chi_1\langle v_x^2\rangle + \xi^2\chi_3\langle v_x^4\rangle) = K_0\left(1 + \xi^2\frac{\chi_3\langle v_x^4\rangle}{\chi_1\langle v_x^2\rangle}\right)$$

$$= K_0\left[1 + \xi^2\frac{24T^2\langle v_x^4\rangle^2}{M^2\langle|\mathbf{v}-\mathbf{v}'|(v_x^2 - v_x'^2)^2\rangle\langle|\mathbf{v}-\mathbf{v}'|(v_x^3 - v_x'^3)^2\rangle}\right]$$

Here K_0 is the ion mobility for the zero electric field.

Let us calculate these terms. We have

$$\langle v_x^4\rangle = \frac{15}{4}\left(\frac{2T}{M}\right)^2 = \frac{15T^2}{M^2}$$

To calculate the denominators, we shall use the dimensionless relative velocity $\mathbf{s} = \sqrt{\frac{M}{2T}}(\mathbf{v} - \mathbf{v}')$ and the dimensionless centre-of-mass velocity $\mathbf{G} = \frac{1}{2}\sqrt{\frac{T}{2M}}(\mathbf{v} + \mathbf{v}')$. We obtain

$$\left(\frac{M}{2T}\right)^{5/2} \langle |\mathbf{v} - \mathbf{v}'| (v_x^2 - v_x'^2)^2 \rangle = 4\langle s s_x^2 G_x^2 \rangle = \frac{4}{9} \langle s^3 \rangle \langle G^2 \rangle = \frac{8\sqrt{2}}{3\sqrt{\pi}}$$

$$\left(\frac{M}{2T}\right)^{7/2} \langle |\mathbf{v} - \mathbf{v}'| (v_x^3 - v_x'^3)^2 \rangle = \left\langle s s_x^2 \left(3G_x^2 + \frac{s_x^2}{4}\right)^2 \right\rangle$$

$$= \frac{3}{5} \langle s^3 \rangle \langle G^4 \rangle + \frac{1}{10} \langle s^5 \rangle \langle G^2 \rangle + \frac{1}{112} \langle s^7 \rangle = \frac{3 \cdot 169\sqrt{2}}{70\sqrt{\pi}}$$

Hence, the ion mobility to the first order of approximation is

$$K = K_0(1 + 6.9\xi^2)$$

Problem 4.10. Find the drift velocity of ions whose mass is much greater than the mass of the gas atoms.

The width of the ion velocity distribution is determined either by the thermal velocity of ions or by the mean variation of the ion velocity in one collision with a gas atom. The thermal velocity of ions is low compared with the thermal velocity of atoms and the mean ion velocity variation is of the order of mg/M; thus, the width of the ion velocity distribution is always smaller than the relative velocity of the ion-atom collision. Therefore, in eq. (1.22) we can make the replacement $f(\mathbf{v}) = \delta(\mathbf{v} - \mathbf{w})$, where $\mathbf{w} = \int f(\mathbf{v})\, \mathbf{v}\, d\mathbf{v}$ is the drift ion velocity. Then we obtain the following expression for the drift ion velocity:

$$\frac{eE}{\mu N} = \left(\frac{m}{2\pi T}\right)^{3/2} \int \exp\left(-\frac{m v_1^2}{2T}\right) g_x g \sigma^*(g)\, d\mathbf{v}_1$$

$$\equiv \frac{1}{w^2} \exp\left(-\frac{m w^2}{2T}\right) \left(\frac{2T}{\pi m}\right)^{1/2} \int_0^\infty \exp\left(-\frac{m g^2}{2T}\right) g^2 \sigma^*(g)\, dg$$

$$\times \left[\frac{mwg}{T} \cosh \frac{mwg}{T} - \sinh \frac{mwg}{T}\right] \tag{4.15}$$

When the ion drift velocity is small compared with the thermal velocity of the gas atoms, we have

$$w = \frac{3\sqrt{\pi}\, eE}{8N\bar{\sigma}\sqrt{2Tm}}, \quad \bar{\sigma} = \int_0^\infty \exp(-x^2)x^5\sigma^*(x)\,dx, \quad x^2 = \frac{mg^2}{2T} \tag{4.16 a}$$

$$\sigma^* = \int (1 - \cos\vartheta)\,d\sigma$$

This result coincides with the first Chapman-Enskog approximation but is valid in a wider velocity range ($w \ll \sqrt{T/m}$) than this approximation ($w \ll \sqrt{T/M}$) . It should be noted that this method involving the replacement of the distribution function with the delta-function is applicable if $w \gg \sqrt{T/M}$. However, the above result is valid for all drift velocities, since for $w \ll \sqrt{T/M}$ the drift velocity is proportional to the electric field and the proportionality factor can be found using this method.

For high drift velocities $w \gg \sqrt{T/M}$ ($w \gg g$) eq. (4.16 a) yields

$$eE/mN = w^2\sigma^*(w) \tag{4.16 b}$$

Equation (4.16 b) follows directly from the motion equation for the ion if we take into account that the variation of its momentum in collisions with the gas atoms is small. Indeed, the motion equation for the ion is

$$\frac{dP_x}{dt} = eE - \int \Delta P_x\, Nw\, d\sigma = eE - mNw^2\sigma^*(w) = 0$$

where P_x is the projection of the ion momentum on the x-axis (parallel to the field), $\Delta \mathbf{P} = m\mathbf{w}(1 - \cos\vartheta)$ is the variation of the ion momentum upon collision with a light atom with a small velocity, and ϑ is the scattering angle.

Problem 4.11. Compare the exact formula for the mobility of electrons in a gas in weak fields with the Chapman-Enskog approximation. Assume that the cross section of elastic scattering of electrons by the atoms depends on the collision velocity as $\sigma^* = Cv^{-k}$.

Equation (2.37) yields for the mobility of electrons in weak fields

$$K = \frac{w}{E} = \frac{8e}{3\sqrt{\pi m}} \int_0^\infty \frac{t^4 \exp(-t^2)\,dt}{\nu_{el}}, \quad t = \sqrt{\frac{mv^2}{2T}}$$

where ν_{el} is the frequency of elastic collisions between electrons and the gas atoms.

Comparing this equation with the first Chapman-Enskog approximation (4.3) for the given dependence of the diffusion cross section σ^* on the collision velocity, we obtain

$$\frac{K}{K_{\mathrm{I}}} = \frac{16}{9\pi}\Gamma\left(3 - \frac{k}{2}\right)\Gamma\left(2 + \frac{k}{2}\right)$$

It should be noted that in practice $0 < k < 2$. In the case of the Coulomb interaction between particles ($k = 4$) collisions involve many particles, while the above equations have been derived for two-particle collisions.

Problem 4.12. Find the dependence of the ion drift velocity on the electric field for strong fields. Assume that the scattering of ions by atoms is described by the classical equations and the potential of interaction between the ions and the atoms in the given interaction energy range is given by $U \sim R^{-n}$, where R is the distance between the nuclei.

In strong electric fields, when $eE/TN\sigma^* \gg 1$, the ion drift velocity is much higher than the thermal velocity of ions and the gas particles. In this case the relative velocity of collision between an ion and a gas particle is equal to the ion velocity and eq. (1.22) can be rewritten as

$$eE/\mu N = \int f(\mathbf{v})\mathbf{v}v_x\sigma^*(v)\,d\mathbf{v}$$

Since in this limiting case the only parameter with the dimension of velocity, determining the distribution function, is the ion drift velocity, we obtain from the above equation $eE/\mu N \sim w^2\sigma^*(w)$. Since the potential of interaction between the particles $U \sim R^{-n}$, the cross section of collision between these particles is related to the collision velocity by $\sigma \sim \mu^{-2/n}v^{-4/n}$ [$\sigma \sim \rho^2$, $U(\rho) \sim \mu v^2$]. Then we obtain

$$w \sim \frac{1}{\sqrt{\mu}}\left(\frac{eE}{N}\right)^{\frac{n}{2(n-2)}}$$

Problem 4.13. Express the ion mobility in a mixture of gases in weak electric fields in terms of the ion mobilities in each of the constituent gases.

The gas mobility (4.3) in the first Chapman-Enskog approximation can be conveniently expressed as

$$1/K = A\nu_{\mathrm{eff}}$$

Here A is a function of gas parameters, independent of the gas species, and ν_{eff} is the effective frequency of collisions between ions and the gas particles, which for a mixture of gases is an additive function of the frequencies of collisions with the particles of each of the gas species. Since we calculate the mobility for the constant total number density of particles and the frequency of collisions with the particles of the ith gas species is $\nu_i = N_i\langle v\sigma_i\rangle$ (where $N_i = c_i N$ and c_i is the concentration of the ith gas species) we have $\nu_{\text{eff}} = \sum_i c_i \nu_i$. Since the mobility of ions in the ith gas species is given by $1/K_i = A\nu_i$, the mobility of ions in the mixture of gases is found from

$$1/K = \sum_i c_i/K_i$$

This relation is known as the *Blanck law* and it has been derived only for weak electric fields. Under these conditions it has the same accuracy as the Chapman-Enskog approximation.

Problem 4.14. Calculate the mean ion energy in a gas in the constant electric field if the frequency of collisions between ions and the gas particles is independent of the collision velocity.

Multiply the kinetic equation (1.5) for ions by the ion energy $Mv^2/2$ and integrate the result over the ion velocities. Since $\int v^2 \frac{\partial f}{\partial v_x} d\mathbf{v} = -2\int f \frac{\partial v^2}{\partial v_x} d\mathbf{v} = -2w$, where w is the mean ion velocity, we obtain

$$eEw = \int \frac{M}{2}(v^2 - v'^2) f(\mathbf{v})\varphi(v_1) Ng\, d\sigma\, d\mathbf{v}\, d\mathbf{v}_1$$

where $\mathbf{v}$ and $\mathbf{v}'$ are the ion velocities before and after the collision, $\mathbf{v}_1$ is the velocity of the gas particle, $\mathbf{g} = \mathbf{v} - \mathbf{v}_1$ is the relative collision velocity, N is the number density of the gas particles, $d\sigma$ is the differential cross section of scattering, $f(\mathbf{v})$ is the ion velocity distribution function, $\varphi(v)$ is the Maxwellian velocity distribution for atoms and both distribution functions are normalized to unity $\left[\int f(\mathbf{v})\, d\mathbf{v} = \int \varphi(v)\, d\mathbf{v} = 1\right]$.

The above integral relation can be used to find the mean ion energy. First, integrate the left-hand side over the scattering angle. The ion velocity is

$$\mathbf{v} = \mathbf{V} + \frac{m}{m+M}\mathbf{g}$$

where $\mathbf{V}$ is the centre-of-mass velocity, and m is the mass of the gas particle. Hence we obtain

$$v^2 - v'^2 = \frac{2m}{M+m}\mathbf{V}(\mathbf{g} - \mathbf{g}')$$

where $\mathbf{g}'$ is the relative velocity of the particles after scattering. Further, we have $\mathbf{g}' = \mathbf{g}\cos\theta + \mathbf{n}g\sin\theta$, where θ is the scattering angle, and $\mathbf{n}$ is the unit vector perpendicular to the vector $\mathbf{g}$. Averaging over the azimuthal scattering angle yields

$$\int(\mathbf{g} - \mathbf{g}')\,d\sigma = \mathbf{g}\int(1 - \cos\theta)\,d\sigma = \mathbf{g}\sigma^*(g)$$

where $\sigma^*(g) = \int(1 - \cos\theta)\,d\sigma$ is the diffusion cross section of the elastic collisions between ions and the gas particles. Then we can rewrite the initial integral relation as

$$eEw = \mu\int f(\mathbf{v})\varphi(v_1)\,\mathbf{V}\mathbf{g}\nu\,d\mathbf{v}\,d\mathbf{v}_1 \equiv \frac{\mu}{m+M}\langle(Mv^2 - mv_1^2)\nu\rangle \tag{4.17}$$

where $\mu = mM/(m+M)$ is the reduced mass of the ion and the atom, $\nu(g) = Ng\sigma^*(g)$ is the frequency of collisions between ions and the gas atoms, and the angle brackets denote averaging over the velocities of the ions and the gas atoms.

Equation (4.17) is the balance equation for the energy of ions in the gas in the constant electric field. The left-hand side of this equation is the energy acquired by the ion per unit time from the electric field. The right-hand side of the equation is the energy lost by the ion per unit time in collisions with the gas atoms. According to eq. (4.7), $w = eE/\mu\nu$ and since the collision frequency ν is independent of the velocity, we obtain

$$w^2 = \int f(\mathbf{v})\varphi(v_1)\mathbf{V}\mathbf{g}\,d\mathbf{v}\,d\mathbf{v}_1 = \langle\mathbf{V}\mathbf{g}\rangle$$

Further, we have $\mathbf{V} = (M\mathbf{v} + m\mathbf{v}_1)/(M+m)$ and $\mathbf{g} = \mathbf{v} - \mathbf{v}_1$, so that

$$w^2 = \langle\mathbf{V}\mathbf{g}\rangle = \frac{M}{m+M}\langle v^2\rangle - \frac{m}{m+M}\langle v_1^2\rangle + \frac{M-m}{M+m}\langle\mathbf{v}\rangle\langle\mathbf{v}_1\rangle$$

Since the mean velocity of the gas atoms is zero, we obtain finally

$$\bar{\varepsilon} = \frac{(m+M)w^2}{2} + \frac{3}{2}T \tag{4.18}$$

where $\bar{\varepsilon} = \left\langle\frac{Mv^2}{2}\right\rangle$ is the mean ion energy and $T = \frac{2}{3}\langle mv_1^2\rangle$ is the gas temperature.

Problem 4.15. Find the mean translational energy for the ion motion in the direction of the external electric field in a gas in a constant field. The frequency of collisions between ions and the gas particles is assumed to be independent of the collision velocity.

We shall use the method of Problem 4.14 and find the mean ion energy from the integral relation. Multiply the kinetic equation for the ion velocity distribution function by the ion kinetic energy $Mv_x^2/2$ for the motion in the direction of the field. We obtain

$$eEw = \frac{M}{2}\int (v_x^2 - v_x'^2) f(\mathbf{v})\varphi(v_1) N g \, d\sigma \, d\mathbf{v} \, d\mathbf{v}_1$$

where $\mathbf{v}$ and $\mathbf{v}'$ are the ion velocities before and after the collision, $\mathbf{v}_1$ is the velocity of the gas particles, $\mathbf{g} = \mathbf{v} - \mathbf{v}_1$ is the relative collision velocity, M is the ionic mass, w is the mean ion velocity in the direction of the field (the x-axis is taken parallel to the field), and $d\sigma$ is the differential cross section of the collision between an ion and a gas atom. The velocity distributions for ions $f(\mathbf{v})$ and gas particles $\varphi(v_1)$ are normalized to unity.

Let us transform the right-hand side of the above equation. Write the ion velocity as

$$\mathbf{v} = \mathbf{V} + \frac{m}{m+M}\mathbf{g}$$

where m is the mass of the gas particle, and $\mathbf{V}$ is the velocity of the centre of mass of the ion and the gas particle. Then we obtain

$$v_x^2 - v_x'^2 = \frac{2m}{m+M} V_x(g_x - g_x') + \left(\frac{m}{m+M}\right)^2 (g_x^2 - g_x'^2)$$

The relative velocity of the ion and the gas particle after collision is

$$\mathbf{g}' = \mathbf{g}\cos\theta + \mathbf{n}g\sin\theta$$

where θ is the scattering angle in the centre-of-mass reference frame, and $\mathbf{n}$ is the unit vector perpendicular to the vector $\mathbf{g}$. Since all the directions of the vector $\mathbf{n}$ in the plane perpendicular to $\mathbf{g}$ have equal probabilities, integration over the scattering angles yields

$$\int (g_x - g_x')\, d\sigma = \int (\mathbf{g} - \mathbf{g}')_x \, d\sigma$$

$$= g_x \int (1 - \cos\theta)\, d\sigma - \overline{n}_x g \int \sin\theta \, d\sigma = g_x \sigma^*$$

Here $\sigma^*(g) = \int (1 - \cos\theta)\, d\sigma$ is the diffusion cross section of collision between the ion and the gas particle and we used the fact that the mean value of the projection of the vector **n** on the x-axis is zero. Denote by ϑ the angle between the vectors **g** and **E** and by φ the angle formed by the projection of the vector **E** and the projection of the vector **g′** in the plane perpendicular to the vector **g** (see Fig. 4.3). We obtain

$$g_x' = g\,(\cos\theta\cos\vartheta + \sin\theta\sin\vartheta\cos\varphi) = g_x\cos\theta + g_\perp\sin\theta\cos\varphi$$

where $g_\perp$ is the projection of the relative velocity **g** on the plane perpendicular to the electric field. Averaging over the azimuthal angle yields

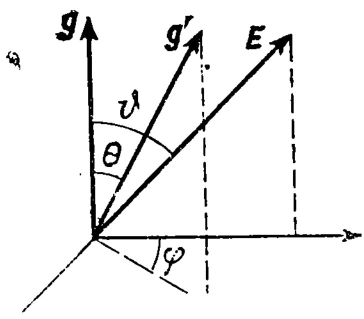

Fig. 4.3. Averaging of vectors in Problem 4.15.

$$g_x^2 - g_x'^2$$

$$= g_x^2 - g_x^2\cos^2\theta - g_\perp^2\sin^2\theta\,\overline{\cos^2\varphi} - 2g_xg_\perp\sin\theta\cos\theta\,\overline{\cos\varphi}$$

$$= \left(g_x^2 - \frac{g_\perp^2}{2}\right)\sin^2\theta = \frac{3}{2}\left(g_x^2 - \frac{g^2}{3}\right)\sin^2\theta$$

where the bar over the symbols denotes averaging over the azimuthal angle and we used the fact that $g_x^2 + g_\perp^2 = g^2$. Then we have

$$\int (g_x^2 - g_x'^2)\,d\sigma = \frac{3}{2}\left(g_x^2 - \frac{g^2}{3}\right)\sigma^2(g)$$

where $\sigma^{(2)}(g) = \int (1 - \cos^2\theta)\,d\sigma$.

Substituting these results into the initial integral relation, we obtain

$$eEw = \mu\langle V_xg_x\nu_1\rangle + \frac{3\mu m}{4(M+m)}\left\langle\left(g_x^2 - \frac{g^2}{3}\right)\nu_2\right\rangle \qquad (4.19)$$

where the angle brackets denote the averaging over the velocities of ions and gas particles, $\nu_1 = Ng\sigma^*(g)$, $\nu_2 = Ng\sigma^{(2)}(g)$, and $\mu = \dfrac{mM}{m+M}$ is the reduced mass of the ion and the gas particle. Now we shall take into account the fact that the frequencies ν_1 and ν_2 of collisions between ions and the gas particles are independent of the collision velocity. Taking the collision frequencies outside the angle brackets and using the velocities of ions and gas particles instead of the relative velocity and the centre-of-mass velocity, we obtain

$$eEw = \mu\nu_1 \frac{(\langle Mv_x^2\rangle - \langle mv_{1x}^2\rangle)}{m+M} + \frac{3\mu m\nu_2}{4(M+m)}\left\langle v_x^2 - \frac{v^2}{3}\right\rangle$$

where we have used the facts that the mean velocity of the gas particles is zero ($\langle v_{1x}\rangle = 0$) and the velocity distribution of the gas particles is isotropic ($\langle v_{1x}^2\rangle = 1/3\,\langle v_1^2\rangle$).

Using eq. (4.7) for the ion drift velocity ($w = eE/\mu\nu_1$), the expression for the translational energy of the atom ($\langle mv_{1x}^2\rangle = T$, where T is the gas temperature) and eq. (4.18) for the mean ion energy $\left(\left\langle\frac{Mv^2}{2}\right\rangle = \frac{(M+m)}{2}w^2 + \frac{3}{2}T\right)$, we obtain the following expression for the mean ion energy associated with the motion in the direction of the electric field:

$$\left\langle\frac{Mv_x^2}{2}\right\rangle = \frac{T}{2} + \frac{(M+m)w^2}{2}\,\frac{\nu_1 + \frac{m}{4M}\nu_2}{\nu_1 + \frac{3m}{4M}\nu_2} \tag{4.20a}$$

Accordingly, the mean ion energy associated with the motion perpendicular to the electric field is

$$\left\langle\frac{Mv_y^2}{2}\right\rangle = \left\langle\frac{Mv_z^2}{2}\right\rangle = \frac{1}{2}\left(\left\langle\frac{Mv^2}{2}\right\rangle - \left\langle\frac{Mv_x^2}{2}\right\rangle\right)$$

$$= \frac{T}{2} + \frac{(M+m)w^2}{2}\,\frac{\frac{m}{4M}\nu_2}{\nu_1 + \frac{3m}{4M}\nu_2} \tag{4.20b}$$

Let us analyze the above results. If the ionic mass M is much smaller than the mass of the gas particle, then the mean kinetic energies of motion along the field and perpendicular to it are the same. This is the case of electron motion in a gas in an external electric field. Elastic scattering of the electron by the gas particle results in a sharp change of its momentum but the variation of the electron energy is small. Therefore, the velocity distribution function for electrons is close to being spherically symmetric and the mean energies of motion in various directions are the same.

If the ionic mass is considerably greater than the mass of the gas particle, the energy of motion of ions perpendicular to the electric field is independent of the field strength if the ion drift velocity is much lower than the thermal velocity of the gas particles. For strong electric fields the mean kinetic energy of ions is determined by their motion along the field.

Problem 4.16. Find the velocity distribution function for ions in a gas in the constant electric field if the ionic mass is much greater than the mass of the gas particle.

Collisions of heavy ions with gas particles do not change much the ion velocity. The ratio of the characteristic variation of the ion velocity in collision to the mean ion velocity can be used as the small parameter for expanding the distribution function. In the zero approximation the ion velocity distribution in terms of the gas particle velocities is the delta function. The width of the function is of the order of the thermal velocity of the ion in the absence of an electric field. In the next approximation of the expansion of the velocity distribution in powers of the above small parameter we can write the function (correct to the normalization factor) in the form of the normal distribution

$$f(\mathbf{v}) = \text{const} \cdot \exp\left[-\frac{M(v_x - w)^2}{2T_{\parallel}} - \frac{Mv_{\perp}^2}{2T_{\perp}}\right] \tag{4.21}$$

where v_x and $v_{\perp}$ are the components of the ion velocity parallel and perpendicular to the electric field, w is the drift ion velocity, M is the ionic mass, and $T_{\parallel}$ and $T_{\perp}$ coincide with the gas temperature T in the absence of the field. Tending $T_{\parallel}$ and $T_{\perp}$ to zero, we see that the ion velocity distribution tends to the delta function $f(\mathbf{v}) = \delta(\mathbf{v} - \mathbf{w})$.

The parameters of the ion velocity distribution function can be found from the mean energies of ion motion in the direction parallel to the field (longitudinal) and perpendicular to the field (transverse). In particular, if the frequency of collisions of ions with the gas particles is independent of the collision velocity, the results of Problem 4.15 yield the following first-order approximations of the expansion in powers of m/M:

$$T_{\parallel} \equiv \langle M(v_x - w)^2 \rangle = T + mw^2\left(1 - \frac{\nu_2}{2\nu_1}\right) \tag{4.22a}$$

$$T_{\perp} \equiv \langle Mv_y^2 \rangle = \langle Mv_z^2 \rangle = T + mw^2 \frac{\nu_2}{4\nu_1} \tag{4.22b}$$

These equations show that if the ion drift velocity is low compared with the thermal velocity of the gas particles, the ion velocity distribution has a Maxwellian form but its centre is shifted and the ion temperature is equal to the gas temperature:

$$f(\mathbf{v}) = \text{const} \cdot \exp\left[-\frac{M(\mathbf{v} - \mathbf{w})^2}{2T}\right] \tag{4.23}$$

If the ion drift velocity is much higher than the thermal velocity of the gas particles, the longitudinal $T_{\parallel}$ and the transverse $T_{\perp}$ temperatures are, generally, different.

Problem 4.17. Find the longitudinal and transverse temperatures corresponding to the velocity distribution function for heavy ions in a gas in an external electric field. The ion drift velocity is much higher than the thermal velocity of the gas particles.

Since the variation of the ion velocity in a single collision with a gas particle is small, the ion velocity distribution has the form of the normal distribution (4.21). The ion velocities are concentrated in a narrow velocity range around the ion drift velocity. We have to find the parameters $T_{\parallel}$ and $T_{\perp}$ in the distribution (4.21), which describe the width of the distribution function.

We shall make use of the integral relations (4.6), (4.17) and (4.19) which can be rewritten for an arbitrary dependence of the collision frequencies on the relative collision velocity as

$$eE = \mu\langle v_x \nu_1\rangle$$

$$eEw = \frac{\mu}{m+M}(\langle Mv^2\nu_1\rangle - \langle mv_1^2\nu_1\rangle) \tag{4.24}$$

$$eEw = \frac{\mu}{m+M}(\langle Mv_x^2\nu_1\rangle - \langle mv_{1x}^2\nu_1\rangle) + \frac{3\mu m}{4\,(M+m)}\left\langle\left(v_x^2 - \frac{v^2}{3}\right)\nu_2\right\rangle$$

The first relation shows that the momentum acquired by the ion per unit time from the electric field is equal to the momentum transferred to the gas particles per unit time; the second relation shows a similar equality for kinetic energy; and the third relation shows a similar equality for the kinetic energy of motion in the direction of the field. In these relations M, m and μ are the ionic mass, the mass of the gas particle and the reduced mass of the ion and the gas particle, respectively; ν_1 and ν_2 are the frequencies of collisions between ions and the gas particles, corresponding to different methods of averaging of the collision cross section over the scattering angles; $\mathbf{v}$ and $\mathbf{v}_1$ are the velocities of the ion and the gas particle; $\mathbf{w}$ is the ion drift velocity; the x-axis is taken parallel to the electric field, and the angle brackets denote averaging over the velocities of ions and the gas particles.

Since the ion drift velocity is much higher than the thermal velocity of the gas particles, the ion temperatures $T_{\parallel}$ and $T_{\perp}$ are much higher than the gas temperature and the thermal motion of the gas particles can be ignored. Subtracting the third relation of (4.24) from the second one, we obtain

$$\langle Mv_{\perp}^2\nu_1\rangle = \frac{3m}{4}\left\langle\left(v_x^2 - \frac{v^2}{3}\right)\nu_2\right\rangle$$

where $v_\perp$ is the velocity component perpendicular to the field. In the zero approximation we can replace the ion velocity distribution by the delta function assuming that the ion velocity coincides with the drift velocity. Then we obtain

$$\langle Mv_\perp^2\rangle = \frac{mw^2}{2}\,\frac{\nu_2}{\nu_1}$$

where the collision frequencies correspond to the relative velocity of collision equal to the ion drift velocity. Hence, the transverse temperature is

$$T_\perp = \langle Mv_y^2\rangle = \langle Mv_z^2\rangle = \frac{1}{2}\langle Mv_\perp^2\rangle = \frac{mw^2}{4}\,\frac{\nu_2}{\nu_1} \qquad (4.25a)$$

To find the longitudinal temperature, rewrite the first relation of (4.24) as

$$eE = \mu\langle v_x\nu_1\rangle = \mu w\langle\nu_1\rangle + \mu\langle(v_x - w)\nu_1\rangle$$

$$= \mu w\langle\nu_1\rangle + \mu\langle(v_x - w)^2\rangle \left.\frac{d\nu_1}{dv}\right|_w$$

Expanding the right-hand side of the second relation of (4.24) in powers of the small parameters $T_\parallel/Mw^2$ and $T_\perp/Mw^2$ and taking the first term of the expansion, we obtain

$$\langle Mv^2\nu_1\rangle$$

$$= \langle Mw^2\nu_1\rangle + 2Mw\,\langle(v_x - w)\nu_1\rangle + \langle M(v_x - w)^2\nu_1\rangle + \langle Mv_\perp^2\nu_1\rangle$$

$$= (Mw^2 + T_\parallel + 2T_\perp)\,\langle\nu_1\rangle + 2wT_\parallel \left.\frac{d\nu_1}{dv}\right|_w$$

Replace the electric field on the left-hand of the second equation of (4.24) by the expression derived above from the first equation of (4.24). After similar transformations the second equation (4.24) yields

$$\mu w^2\langle\nu_1\rangle + \frac{\mu T_\parallel}{M}\,w \left.\frac{d\nu_1}{dv}\right|_{v=w}$$

$$= \frac{\mu}{M+m}\left[(Mw^2 + T_\parallel + 2T_\perp)\langle\nu_1\rangle + 2T_\parallel\, w \left.\frac{d\nu_1}{dw}\right|_{v=w}\right]$$

Hence, we obtain

$$T_\parallel + 2T_\perp - T_\parallel \left.\frac{d(\ln\nu_1)}{d(\ln v)}\right|_{v=w} = mw^2$$

Using eq. (4.25a), we find the longitudinal temperature:

$$T_{\parallel} = mw^2 \frac{(1 - \nu_2/2\nu_1)}{1 + \left.\dfrac{d(\ln \nu_1)}{d(\ln v)}\right|_{v=w}} \tag{4.25b}$$

We see that both temperatures describing the velocity distribution of ions are of the order of mw^2. In this case the drift ion velocity is considerably higher than the thermal velocity of the gas particles, which is of the order of $\sqrt{T/m}$, where T is the gas temperature. This means that the longitudinal and transverse temperatures of ions are considerably higher than the gas temperature.

Equation (4.25a) for the transverse temperature derived for an arbitrary dependence of the collision frequency on the collision velocity coincides with eq. (4.22b) derived for strong fields. Therefore, eq. (4.22b) derived for the constant frequency of collisions between ions and the gas particles is valid in a more general case. As for eq. (4.25b) for the longitudinal temperature, it coincides with eq. (4.22a) only if the collision frequency ν_1 is independent of the relative collision velocity, that is, if the condition of applicability of eq. (4.22a) is satisfied. According to eq. (4.22b), the longitudinal temperature of ions depends on the character of the variation of the collision frequency with the change in the relative collision velocity.

Problem 4.18. Find the drift velocity of ions in a gas in a strong electric field. The ion mass is equal to the mass of the gas particles and the ion drift velocity is much higher than the thermal velocity of gas particles. Assume that the cross section of scattering of the ion by the gas particle is independent of the collision velocity and scattering is isotropic in the centre-of-mass reference frame.

Scattering in this case corresponds to the so-called rigid sphere model in which the potential of interaction between the particles varies sharply in the range of distances between the particles at which scattering occurs. This model describes real scattering events.

To simplify the analysis in this problem we shall use an approximation technique in which we shall select a reasonable form for the ion velocity distribution, and the respective parameters can be found from the integral relations for the mean characteristics of the ions. Clearly, eq. (4.21) gives a suitable expression of the ion velocity distribution since it correctly describes the velocity distribution of ions in the velocity range where they are mainly concentrated. Thus, let us take the ion velocity distribution function in the form

$$f(\mathbf{v}) = C \exp\left[-\frac{M(v_x - w)^2}{2T_{\parallel}} - \frac{Mv_{\perp}^2}{2T_{\perp}}\right]$$

where C is the normalization constant, and the ion drift velocity w and the longitudinal and transverse temperatures $T_{\parallel}$ and $T_{\perp}$ must be determined.

We shall use the integral relations (4.24) to determine the ion parameters. Since scattering of ions by the gas particles is isotropic, we have $\sigma^{(2)} \equiv \equiv \int (1-\cos^2\theta)\, d\sigma = \frac{2}{3}\int d\sigma = \frac{2}{3}\int (1-\cos\theta)\, d\sigma = \frac{2}{3}\sigma^*$, where θ is the scattering angle, and σ^* is the diffusion cross section of scattering. Therefore, when the collision cross section is independent of the collision velocity, we have $\nu_1 \equiv Nv\sigma^* = v/\lambda$ and $\nu_2 \equiv \frac{2}{3}\nu_1 = \frac{2v}{3\lambda}$, where v is the ion velocity and $\lambda = 1/N\sigma^*$ is the ion mean free path in the gas. Using the fact that the ionic mass is equal to the mass of the gas particle and ignoring the velocity of the gas particles, which is much lower than the ion velocity, we can rewrite the integral relations (4.24) in the following form:

$$eE = \frac{m}{2\lambda}\langle v_x v\rangle$$

$$eEw = \frac{m}{4\lambda}\langle v^3\rangle$$

$$eEw = \frac{m}{4\lambda}\langle v_x^2 v\rangle + \frac{m}{8\lambda}\left(\left\langle\left(v_x^2 - \frac{v^2}{3}\right)v\right\rangle\right)$$

As before, the angle brackets denote the averaging over the ion velocities, and m is the mass of the gas particle equal to the ionic mass.

Let us use the above ion velocity distribution for the averaging over the ion velocities. Then we obtain three equations for the three parameters determining the ion velocity distribution.

To simplify calculations, we can make another approximation. In the terms with odd powers of the ion velocity we shall use the following expansion:

$$v = \sqrt{v_x^2 + v_\perp^2} \approx v_x + \frac{v_\perp^2}{2v_x}$$

Thus, we assume that the kinetic energy of the ion motion along the field is much higher than the kinetic energy of the motion perpendicular to the field. The accuracy of this approximation can be checked from the results obtained in solving this problem.

Using the above approximation, we can transform the first integral relation into

$$eE = \frac{m}{2\lambda}\left\langle v_x^2 + \frac{v_\perp^2}{2}\right\rangle = \frac{m}{2\lambda}\left((w^2 + \langle (v_x - w)^2\rangle + \left\langle\frac{v_\perp^2}{2}\right\rangle\right)$$

which yields

$$eE\lambda = \frac{mw^2}{2} + \frac{T_{\parallel}}{2} + \frac{T_{\perp}}{2}$$

The second integral relation gives

$$eEw = \frac{m}{4\lambda}\left(\left\langle (v_x^2 + v_{\perp}^2)\left(v_x + \frac{v_{\perp}^2}{2v_x}\right)\right\rangle\right)$$

$$= \frac{m}{4\lambda}\left(\langle v_x^3\rangle + \frac{3}{2}\, w\,\langle v_{\perp}^2\rangle + \frac{\langle v_{\perp}^4\rangle}{2w}\right)$$

$$= \frac{m}{4\lambda}\left(w^3 + 3w\langle (v_x - w)^2\rangle + \frac{3}{2}\, w\langle v_{\perp}^2\rangle + \frac{\langle v_{\perp}^4\rangle}{2w}\right)$$

After averaging, we obtain

$$eE\lambda = \frac{m}{4}\, w^2 + \frac{3}{4}\, T_{\parallel} + \frac{3}{4}\, T_{\perp} + \frac{T_{\perp}^2}{mw^2}$$

Further, we take the difference between the second and the third integral relations, which is

$$0 = \frac{m}{4\lambda}\,\langle v_{\perp}^2 v\rangle - \frac{m}{8\lambda}\left\langle\left(v_x^2 - \frac{v^2}{3}\right)v\right\rangle$$

This expression yields the following equation for the ion parameters:

$$\frac{m}{4\lambda}\cdot\frac{2T_{\perp}w}{m}\left(1 + \frac{2T_{\perp}}{mw^2}\right) - \frac{m}{8\lambda}\cdot\frac{2}{3}\, w^3\left(1 + \frac{3T_{\parallel}}{mw^2} - \frac{2T_{\perp}^2}{m^2w^4}\right) = 0$$

or

$$\frac{mw^2}{6} + \frac{1}{2}\, T_{\parallel} - T_{\perp} - \frac{7}{3}\,\frac{T_{\perp}^2}{mw^2} = 0$$

Let us define new variables $x = mw^2/eE\lambda$, $y = T_{\parallel}/eE\lambda$ and $z = T_{\perp}/eE\lambda$. We have the following system of equations for these variables:

$$1/2(x + y + z) = 1$$

$$\frac{x}{4} + \frac{3}{4}(y + z) + \frac{z^2}{x} = 1$$

$$\frac{x}{6} + \frac{y}{2} = z + \frac{7}{3}\,\frac{z^2}{x}$$

The solution of this system of equations is $x = 1.15$, $y = 0.555$ and $z = 0.294$, so that the ion parameters are

$$w = 1.07\sqrt{\frac{eE\lambda}{m}}, \quad T_{\parallel} = 0.555eE\lambda, \ T_{\perp} = 0.294eE\lambda$$

The exact solution of the problem is

$$w = 1.14\sqrt{\frac{eE\lambda}{m}}, \quad T_{\parallel} = 0.454eE\lambda, \quad T_{\perp} = 0.293eE\lambda$$

The accuracy of the approximation used here is shown by the small difference between the exact and approximated results. Note that if we substitute the above accurate values of the ion parameters into the system of equations, the difference between the left-hand and right-hand sides of each equation is not more than 5%.

§ 4.2. DIFFUSION OF CHARGED PARTICLES IN A GAS IN EXTERNAL FIELDS

Problem 4.19. Proceeding from the kinetic equation, derive the diffusion coefficient for charged particles in a gas in external fields.

Assume that in a weakly ionized gas the number density of charged particles varies spatially and the variation of the number density at distances of the order of the mean free path is small. The gradient of the number density of charged particles gives rise to the diffusion flux of charged particles, which, by definition, is

$$\mathbf{j}_{\text{dif}} = -\mathscr{D}\nabla N_i \tag{4.26}$$

where N_i is the number density of charged particles, which is small compared with the number density of neutral gas particles, and $\mathscr{D}$ is the diffusion coefficient for charged particles in the gas.

We shall expand the kinetic equation in powers of the small parameter $\lambda \frac{\partial(\ln N_i)}{\partial z}$, where λ is the mean free path of charged particles in the gas and the number density of charged particles varies in the direction z. We have

$$\frac{\partial f}{\partial t} + \mathbf{v}\nabla f + \frac{e\mathbf{E}}{M}\frac{\partial f}{\partial \mathbf{v}} = I_c(f)$$

where I_c is the collision integral. The velocity distribution for charged particles is proportional to N_i which depends on the parameter $\mathbf{r} - \mathbf{w}t$, where $\mathbf{r}$ is the coordinate and $\mathbf{w}$ is the drift velocity of the charged particle.

Since the dependence of the velocity distribution on coordinates and time is determined by N_i, we can rewrite the kinetic equation as

$$(\mathbf{v} - \mathbf{w}) \frac{\partial (\ln N_i)}{\partial z} f + \frac{e\mathbf{E}}{M} \frac{\partial f}{\partial \mathbf{v}} = I_c(f)$$

Expand the velocity distribution in powers of the small parameter related to the density gradient:

$$f(\mathbf{v}) = f^{(0)}(\mathbf{v}) - \frac{\partial (\ln N_i)}{\partial z} \Phi(\mathbf{v})$$

Here the function $f^{(0)}(\mathbf{v})$ corresponds to a constant number density of charged particles and is the solution of the kinetic equation

$$\frac{e\mathbf{E}}{M} \frac{\partial f^{(0)}}{\partial \mathbf{v}} = I_c(f^{(0)})$$

The normalization condition for the distribution functions $f(\mathbf{v})$ and $f^{(0)}(\mathbf{v})$ gives the following condition for the function Φ:

$$\int \Phi \, d\mathbf{v} = 0 \qquad (4.26a)$$

The function Φ satisfies the following equation:

$$(v_z - w_z) f^{(0)} = \frac{e\mathbf{E}}{M} \frac{\partial \Phi}{\partial \mathbf{v}} - I_c(\Phi) \qquad (4.27)$$

Here we assumed that the number density of charged particles is low and the collisions between them can be ignored. Therefore, the collision integral is a linear functional of the distribution function for charged particles.

The flux of charged particles, by definition, is $\mathbf{j} = \int \mathbf{v} f \, d\mathbf{v}$ and the diffusion flux due to the density gradient is taken into account by a correction for the equilibrium distribution function. Comparing the expression for the diffusion flux with its definition (4.26), we can find the diffusion coefficient

$$\mathcal{D}_z = \frac{1}{N_i} \int v_z \Phi(\mathbf{v}) \, d\mathbf{v}$$

Using the condition (4.26a), we can write the diffusion coefficient in the following convenient form:

$$\mathcal{D}_z = \frac{1}{N_i} \int (v_z - w) \Phi(\mathbf{v}) \, d\mathbf{v} \qquad (4.28)$$

Problem 4.20. Derive the Einstein relation for charged particles from the kinetic equation.

In weak electric fields the velocity distribution function for charged particles has the form $f = \varphi_0 + v_x\varphi_1$, where φ_0 is the Maxwellian distribution and the x-axis is parallel to the field. Applying the perturbation theory to the kinetic equation

$$\frac{e\mathbf{E}}{m}\frac{\partial f}{\partial \mathbf{v}} = I_c(f)$$

we obtain the equation for φ_1,

$$\frac{e\mathbf{E}}{m}\frac{d\varphi_0}{dv} \equiv \frac{eEv_x}{T}\varphi_0 = I_c(v_x\varphi_1)$$

which is equivalent to eq. (4.27) for $E \to 0$ if we replace φ_1 by $\frac{T}{eE}f_1$. Comparing the drift velocity $w = \frac{1}{N}\int v_x^2\varphi_1\, d\mathbf{v}$ with the expression (4.28) for the diffusion coefficient, we obtain $\mathscr{D} = \frac{T}{eE}w$. Since $w = KE$, this leads to the Einstein relation (1.79) for the mobility K and the diffusion coefficient $\mathscr{D}$ in weak fields:

$$K = e\mathscr{D}/T \tag{4.29}$$

Problem 4.21. Calculate the diffusion coefficient for electrons migrating in a gas perpendicular to the external electric field.

Take the x-axis parallel to the electric field and the z-axis in the direction of the gradient of the electron number density, that is, in the direction of electron diffusion. The electron velocity distribution for the constant electron number density can be approximated by

$$f^{(0)}(\mathbf{v}) = f_0(v) + v_x f_1(v)$$

using the ratio between the mass of the electron and the mass of the gas particle as the small parameter. In a similar way, the function $\Phi(\mathbf{v})$ describing the diffusion of electrons can also be approximated as

$$\Phi(\mathbf{v}) = \Phi_0(v) + v_x\Phi_1(v)$$

Substituting these expressions into eq. (4.27) for $\Phi(\mathbf{v})$, we obtain

$$v_z f_0 + v_z v_x f_1 = \frac{eE}{m}\frac{v_x}{v}\frac{d\Phi_0}{dv} + \frac{eE}{m}\frac{v_z v_x}{v}\frac{d\Phi_1}{dv} - I_c(\Phi_0) + \nu v_z\Phi_1$$

where m is the electron mass. Here we have used the expression for the collision integral of the asymmetric part of the distribution function: $I_c[v_z f(v)] = -\nu v_z f(v)$, where ν is the frequency of collisions between electrons and gas particles.

Multiplying the above equation by v_z and averaging the result over the angles, we obtain the simple expression $f_0 = \nu \Phi_1$, Substituting it into the expression for the diffusion coefficient, we obtain

$$\mathcal{D}_\perp = \frac{1}{N_i} \int v_z^2 \frac{f_0}{\nu} \, d\mathbf{v} = \left\langle \frac{v^2}{3\nu} \right\rangle \tag{4.30}$$

where the angle brackets denote the averaging over the electron velocities.

Problem 4.22. Find the dependence of the diffusion coefficient for ions in a gas on the gas temperature for weak electric fields. The scattering of ions by the gas particles is described by classical equations. The potential of interaction between a test particle and a gas particle for the separations R, for which the potential is of the order of the temperature, can be approximated by $U(R) = CR^{-n}$.

The diffusion coefficient of ions is $\mathcal{D} \sim \overline{v\lambda}$, where $\bar{v} \sim \sqrt{T}$ is the characteristic velocity of ions, $\lambda \sim 1/N\sigma$ is the mean free path of ions, N is the number density of the gas particles, and σ is the cross section of scattering of ions by the gas particles. Further, we have $\sigma \sim \rho^2$ and $U(\rho) \sim \mu \overline{v^2}$, where μ is the reduced mass of the ion and the gas particle. For the given interaction potential we obtain $\sigma \sim T^{-2/n}$ and

$$\mathcal{D} \sim \frac{1}{N} T^{1/2+2/n} \tag{4.31}$$

Thus, the diffusion coefficient varies with the gas temperature as $T^{1/2+2/n}$ if the number density of the gas particles is constant and as $T^{3/2+2/n}$ if the gas pressure is constant.

Problem 4.23. Derive the relationship between the diffusion coefficient and the mobility of electrons in a gas for an arbitrary electric field.

Let us define a parameter η describing the ratio of the diffusion coefficient of electrons in a gas to their mobility:

$$\eta = \frac{eE\mathcal{D}_\perp}{Tw} = \frac{e\mathcal{D}_\perp}{TK}$$

Here E is the electric field strength, and T is the gas temperature. The parameter η is known as the *Townsend coefficient* and for weak electric fields it is unity according to the Einstein relation (4.29). Using the expression

(4.30) for the diffusion coefficient and the electron drift velocity w (2.28a), we obtain

$$\eta = \frac{\left\langle \dfrac{mv^2}{\nu} \right\rangle}{T\left\langle \dfrac{1}{v^2}\dfrac{d}{dv}\left(\dfrac{v^3}{\nu}\right)\right\rangle} \tag{4.32}$$

For instance, if the frequency ν of elastic collisions between electrons and atoms is independent of velocity, eq. (4.32) yields

$$\eta = \left\langle \frac{mv^2}{3T} \right\rangle$$

If the electron velocity distribution has the Maxwellian form, we have

$$\eta = T_e/T$$

In the above cases the Townsend coefficient is the ratio of the mean kinetic energy of electrons to the mean kinetic energy of the gas atoms. This is approximately true in the general case. For instance, for strong electric fields, when the velocity distribution for electrons can be described by the Druyvesteyn function and the cross section of collision between the electrons and the gas particles is independent of the electron velocity (that is, $\nu \sim v$), the Townsend coefficient is $\eta = 1.14\left\langle \dfrac{mv^2}{3T} \right\rangle$. When $\nu \sim v^{2/3}$, we have $\eta = 1.07\left\langle \dfrac{mv^2}{3T} \right\rangle$ and when $\nu \sim v^{1/2}$, we have $\eta = 1.04\left\langle \dfrac{mv^2}{3T} \right\rangle$. We see that the Townsend coefficient gives a sufficiently accurate estimate of the mean electron energy.

Problem 4.24. Find the diffusion coefficient for ions in a gas if the frequency of collisions between ions and the gas particles is independent of the collision velocity.

Let us make use of eq. (4.27) for the function Φ which determines the coefficient of diffusion of ions in a gas. Multiply eq. (4.27) by the ion velocity v_z in the direction of the ion number density gradient and integrate the result over the ion velocities. Taking into account the condition (4.26a), we obtain the following integral relation:

$$(\langle v_z^2 \rangle - w_z^2)N_i = -\int v_z I_c(\Phi)\, d\mathbf{v}$$

where the angle brackets denote the averaging over the ion velocities and N_i is the number density of ions. This relation can be used to obtain the diffusion coefficient.

Let us transform the right-hand side of the integral relation

$$-\int v_z I_c(\Phi)\,d\mathbf{v}$$

$$=\int v_z[\Phi(\mathbf{v})\varphi(v_1)-\Phi(\mathbf{v}')\varphi(v_1')]|\mathbf{v}-\mathbf{v}_1|\,d\sigma\,d\mathbf{v}\,d\mathbf{v}_1$$

$$=\int (v_z-v_z')\,\Phi(\mathbf{v})\varphi(v_1)\,|\,\mathbf{v}-\mathbf{v}_1\,|\,d\sigma\,d\mathbf{v}\,d\mathbf{v}_1$$

Here $\mathbf{v}$ and $\mathbf{v}'$ are the ion velocities before and after the collision, v_1 and v_1' are the velocities of the gas particle before and after the collision, and $d\sigma$ is the differential cross section of scattering. The integrand was transformed by reversing the time and, hence, the velocities before and after the collision. According to the principle of detailed balancing, $|\mathbf{v}-\mathbf{v}_1|\,d\sigma$ is invariant for such a reversal.

Integrate the above relation over scattering angles as was done several times before (see, for instance, Problem 1.10). We obtain

$$\int (v_z-v_z')\,d\sigma=\frac{m}{m+M}g_z\sigma^*$$

where M is the ionic mass, m is the mass of the gas particle, σ^* is the diffusion cross section of collision between the ion and the gas particle, and $\mathbf{g}$ is the relative velocity of collision. Then the initial relation can be rewritten as

$$(\langle v_z^2\rangle - w_z^2)N_i=\frac{m}{m+M}\int g_z\Phi(\mathbf{v})\varphi(v_1)\,|\mathbf{v}-\mathbf{v}_1|\sigma^*\,d\mathbf{v}\,d\mathbf{v}_1 \qquad (4.33)$$

Since the frequency of collisions between ions and the gas particles $\nu=Ng\sigma^*(g)$ is independent of the collision velocity (here N is the number density of gas particles), we obtain

$$(\langle v_z^2\rangle - w_z^2)N_i=\frac{\nu m}{N(m+M)}\int (v_z-v_{1z})\Phi(\mathbf{v})\,d\mathbf{v}\,\varphi(v_1)\,d\mathbf{v}_1$$

The variables on the right-hand side of this equation can be separated and by definition, for gas particles we have $\int\varphi(v_1)d\mathbf{v}_1=N$ and $\int v_{1z}\varphi(v_1)\,d\mathbf{v}_1=0$. Using the integral of the function Φ from the eq. (4.28) for the diffusion coefficient, we obtain

$$\mathscr{D}_z=\frac{M(\langle v_z^2\rangle - w_z^2)}{\mu\nu} \qquad (4.34)$$

where $\mu = mM/(m + M)$ is the reduced mass of the ion and the gas particle. A similar result was obtained in Problem 1.11.

Let us make use of eqs. (4.20) for the mean ion energy under the conditions of the problem. Then we obtain the following expressions for the diffusion coefficients $\mathscr{D}_{\parallel}$ and $\mathscr{D}_{\perp}$ in the directions parallel to the field and perpendicular to it:

$$\mathscr{D}_{\parallel} = \frac{1}{\mu\nu_1}\left[T + mw^2 \frac{\left(\nu_1 - \dfrac{1}{2}\nu_2 + \dfrac{m}{4M}\nu_2\right)}{\left(\nu_1 + \dfrac{3m}{4M}\nu_2\right)}\right]$$

$$\mathscr{D}_{\perp} = \frac{1}{\mu\nu_1}\left[T + mw^2 \frac{\left(1 + \dfrac{m}{M}\right)\nu_2}{4\left(\nu_1 + \dfrac{3m}{4M}\nu_2\right)}\right]$$

Here the frequencies $\nu_1 = Ng\int(1 - \cos\theta)\, d\sigma$ and $\nu_2 = Ng\int(1 - \cos^2\theta)\, d\sigma$ are described by different types of averaging over the scattering angle θ, and w is the ion drift velocity.

Problem 4.25. Calculate the longitudinal and transverse diffusion coefficients for heavy ions in a gas.

We shall calculate the diffusion coefficient in the two limiting cases when the ion drift velocity is much lower or higher than the thermal velocity of the gas particles. Since the width of the velocity distribution is small compared with the characteristic velocities, we can make use of the integral relation for the function Φ determining the diffusion coefficient (4.28).

Multiply eq. (4.27) for the function Φ by the velocity component v_z in the direction of diffusion and integrate the result over ion velocities. Taking into account the condition (4.26a), we obtain finally eq. (4.33):

$$(\langle v_z^2 \rangle - w_z^2)\, N_i = \frac{m}{m + M}\int g_z \Phi(\mathbf{v})\varphi(v_1) g\sigma^*\, d\mathbf{v}\, d\mathbf{v}_1$$

Here the angle brackets denote the averaging over ion velocities, N_i is the number density of ions, $\mathbf{v}$ is the ion velocity, $\mathbf{v}_1$ is the velocity of the gas particle, $\mathbf{g} = \mathbf{v} - \mathbf{v}_1$ is the relative velocity of the ion and the gas particle, $\varphi(v_1)$ is the Maxwellian velocity distribution of gas particles, σ^* is the diffusion cross section of the collision between the ion and the gas particle, M is the ionic mass, and m is the mass of the gas particle.

Substituting $g_z = v_z - v_{1z}$ and using the fact that the rest of the integrand is an even function in v_1, we can transform the above relation into

$$(\langle v_z^2\rangle - w_z^2)N_i = \frac{m}{m+M}\int v_z\Phi(\mathbf{v})\varphi(v_1)g\sigma^*\,d\mathbf{v}\,d\mathbf{v}_1$$

which can be used to find the ion diffusion coefficient.

The function Φ is the solution of the integro-differential equation (4.27) and has a form similar to that of the ion velocity distribution $f^{(0)}$. Therefore, the range of ion velocities for which this function is near its maximum is of the order of the width of the ion velocity distribution. Hence, the integral on the right-hand side of the above equation converges in the narrow velocity range for which the function $\Phi(\mathbf{v})$ differs from zero. Therefore, we can expand the rest of the integrand in this velocity range.

Let us consider at first the transverse diffusion of ions. In weak fields the transverse velocity of ions is small compared with the velocities of the gas particles and in strong fields it is small compared with the drift velocity, too. Thus, we can integrate over velocities of the gas particles, obtaining

$$\langle v_y^2\rangle N_i = \frac{m}{m+M}\int v_y\Phi(\mathbf{v})\,d\mathbf{v}\bar{\nu}$$

where the y-axis is perpendicular to the electric field, and $\bar{\nu}$ is the frequency of collisions between ions and the gas particles, averaged over velocities of the ions and the gas particles. If the ion drift velocity is small compared with the thermal velocity of the gas particles, we have

$$\bar{\nu} = \langle N v_1\,\sigma^*(v_1)\rangle$$

where the angle brackets denote averaging over velocities of the gas particles. In the other limiting case we have

$$\bar{\nu} = Nw\,\sigma^*(w)$$

where w is the ion drift velocity. In any case, $\bar{\nu}$ is independent of the transverse ion velocity, since the latter is small, so that eq. (4.28) yields

$$\mathscr{D}_\perp = \frac{m+M}{m}\,\frac{\langle v_y^2\rangle}{\bar{\nu}} = \frac{T_\perp}{\mu\bar{\nu}}$$

where $T_\perp$ is the ion transverse temperature (4.25), and μ is the reduced mass of the ion and the gas particle.

When the ion drift velocity is small compared with the thermal velocity of gas particles, a similar expression can be derived for the longitudinal diffusion coefficient:

$$\mathscr{D}_\parallel = \frac{(m+M)}{m}\,\frac{((\langle v_x^2\rangle - w^2)}{\bar{\nu}} = \frac{T_\parallel}{\mu\bar{\nu}}$$

Here the ion longitudinal temperature $T_{\|}$ is equal to the gas temperature T.

In the other limiting case, when $w \gg \sqrt{T/m}$, we have

$$(\langle v_x^2\rangle - w^2)N_i = \int v_x \Phi(\mathbf{v})\, d\mathbf{v}\, \nu(v)$$

since the relative velocity of collision between the ion and the gas particle here coincides with the ion velocity v which is close to the ion drift velocity w. Let us expand the quantity $v_x\nu(v)$ in powers of $v_x - w$. Assuming $\nu \sim v^{\gamma}$, that is $\gamma = \left.\dfrac{d(\ln \nu)}{d(\ln v)}\right|_{v=w}$, we obtain

$$v^2 = (w + v_x - w)^2 + v_{\perp}^2 = w^2\left[1 + \frac{2(v_x - w)}{w}\right]$$

so that

$$\nu(v) = \nu(w)\left[1 + \gamma\frac{(v_x - w)}{w}\right]$$

Finally, we have

$$v_x\nu(v) = w\nu(w) + (\gamma + 1)(v_x - w)\nu(w)$$

According to the condition (4.26a), the first term of the above relation gives zero after integration, and using eq. (4.28) we find

$$\mathscr{D}_{\|} = \frac{(m + M)(\langle v_x^2\rangle - w^2)}{m(1 + \gamma)\nu(w)} = \frac{T_{\|}}{\mu\nu(w)(1 + \gamma)}$$

Problem 4.26. Calculate the longitudinal diffusion coefficient for atomic ions in the parent gas for strong electric fields. Assume that the cross section of resonance charge transfer from the ion to the atom is independent of the collision velocity.

The kinetic equation for the ion velocity distribution in this case is given by eq. (4.13). Eliminating the transverse velocity components of ions and atoms, we can rewrite this equation as

$$\frac{eE}{M}\frac{\partial f}{\partial v_x} = \int [f(v_x')\varphi(v_x) - f(v_x)\varphi(v_x')]\, |v_x - v_x'|\sigma_{\text{res}}\, dv_x'$$

where σ_{res} is the cross section of resonance charge transfer from the ion to the atom, and $\varphi(v_x)$ is the Maxwellian velocity distribution for the gas atoms. Since the characteristic velocity of ions is much higher than the velocity of atoms, we can express the atom velocity distribution in terms of the ion velocities as $\varphi(v_x) = N\delta(v_x)$, where N is the number density of atoms. Then we can transform the kinetic equation into

$$\frac{eE}{M}\frac{\partial f}{\partial v_x} = \frac{1}{\lambda}\delta(v_x)\int v'_x f(v'_x)\,dv'_x - \frac{v_x}{\lambda}f(v_x)$$

where $\lambda = (N\sigma_{\text{res}})^{-1}$ is the mean free path of ions in the parent gas. The solution of this equation is eq. (4.12) which was derived in a different way.

Using this kinetic equation, we can write the following equation for the function $\Phi(v_x)$ determining the ion diffusion coefficient using eq. (4.27):

$$(v_x - w)f(v_x) = \frac{eE}{M}\frac{d\Phi}{dv_x} + \frac{v_x}{\lambda}\Phi - \frac{\delta(v_x)}{\lambda}\int v'_x\Phi(v'_x)\,dv'_x$$

Rewrite the equation using the dimensionless variable

$$u = \left(\frac{Mv_x^2}{eE\lambda}\right)^{1/2} = \left(\frac{MN\sigma_{\text{res}}}{eE}\right)^{1/2} v_x$$

Employing the ion drift velocity $w = (2eE/\pi MN\sigma_{\text{res}})^{1/2}$ and eq. (4.12) for the ion velocity distribution, we obtain

$$\frac{2\lambda}{\pi w}N_i\left(u - \sqrt{\frac{2}{\pi}}\right)\exp(-u^2/2)\,\eta(u) = \frac{d\Phi}{du} + u\Phi - \delta(u)\int u'\Phi(u')\,du'$$

where N_i is the number density of ions. The solution of this equation is

$$\Phi = \frac{2\lambda}{\pi w}N_i\exp(-u^2/2)\left(\frac{u^2}{2} - \sqrt{\frac{2}{\pi}}\,u + C\right)\eta(u)$$

The constant C can be found from the condition (4.26a) which should be satisfied by the function Φ: $\int \Phi(u)\,du = 0$. We obtain $C = \frac{2}{\pi} - \frac{1}{2}$, so that

$$\Phi(u) = \frac{2\lambda}{\pi w}\cdot N_i\exp(-u^2/2)\left(\frac{u^2}{2} - \sqrt{\frac{2}{\pi}}\,u + \frac{2}{\pi} - \frac{1}{2}\right)$$

and the longitudinal diffusion coefficient, according to eq. (4.28), is

$$\mathscr{D}_{\parallel} = \frac{1}{N_i}\int (v_x - w)\Phi\,dv_x = \frac{1}{N_i}\cdot\frac{eE\lambda}{M}\int \Phi(u)u\,du$$

$$= \lambda w\left(\frac{2}{\pi} - \frac{1}{2}\right) = 0.137\,w\lambda \qquad (4.35)$$

Problem 4.27. Find the transverse diffusion coefficient for ions in the parent gas in strong electric field.

Take the z-axis perpendicular to the electric field. According to the results of Problem 4.7, the collision integral of the ion distribution $F(v_z)$ over the transverse velocity components v_z is

$$I_c(F) = \varphi(v_z)\int\langle|\mathbf{v} - \mathbf{v}'|\rangle\,\sigma_{\text{res}}F(v_z')\,dv_z' - F(v_z)\int\langle|\mathbf{v} - \mathbf{v}'|\rangle\sigma_{\text{res}}\varphi(v_z')\,dv_z'$$

where the angle brackets denote the averaging over the x- and y-components of the ion and atom velocities, and $\varphi(v_z)$ is the Maxwellian velocity distribution. Since the ion velocity along the field is considerably higher than the characteristic atom velocity and the characteristic transverse ion velocity, we have $\langle|\mathbf{v} - \mathbf{v}'|\rangle = w$, where w is the ion drift velocity. Hence we obtain

$$I_c(F) = \varphi(v_z)w\sigma_{\text{res}}\int F(v_z')\,dv_z' - F(v_z')Nw\sigma_{\text{res}}$$

Using the expression for the collision integral, we can rewrite eq. (4.27) as

$$v_z\varphi(v_z) = Nw\sigma_{\text{res}}\Phi(v_z) - \varphi(v_z)w\sigma_{\text{res}}\int\Phi(v_z)\,dv_z$$

According to the condition (4.26a), the integral in the above equation is zero and the solution of this equation is

$$\Phi(v_z) = \frac{v_z\varphi(v_z)}{Nw\sigma_{\text{res}}}$$

Using eq. (4.28), we finally obtain

$$\mathcal{D}_\perp = \frac{\langle v_z^2\rangle}{Nw\sigma_{\text{res}}} = \frac{T}{Nw\sigma M_{\text{res}}} \tag{4.36}$$

Problem 4.28. Find the coefficient of diffusion of electrons parallel to an electric field. Assume that the mean electron energy is considerably higher than the thermal energy of the gas particles, the frequency of collisions between electrons and the gas particles is independent of the collision velocity, and the mean electron energy is determined only by elastic collisions with the gas particles.

We shall make use of the expressions for the diffusion coefficient and the velocity distribution for electrons. The electron velocity distribution, as shown in Problem 2.12, can be approximated by the following expansion in spherical harmonics:

$$f = f_0(v) + v_x f_1(v)$$

where $f_0(v) = A \exp(-3mv^2/2Mw^2)$ and $f_1(v) = 3mf_0(v)/Mw$ (here A is a normalization factor, w is the electron drift velocity, and M is the mass of the gas particle). The other part of the electron velocity distribution, describing the diffusion motion of electrons, can also be expanded in spherical harmonics:

$$\Phi(\mathbf{v}) = \Phi_0(v) + v_x \Phi_1(v)$$

Substitute the above expansions into eq. (4.27) for $\Phi(v)$, and average it over the angle ϑ between the velocity and the field. Then multiply this equation by $\cos\vartheta = v_x/v$ and average the result over the angle. We thus derive the system of two equations for the functions Φ_0 and Φ_1:

$$\frac{v^2}{3} f_1 - wf_0 = \frac{eE}{m}\left(\Phi_1 + \frac{v}{3}\frac{d\Phi_1}{dv}\right) - I_c(\Phi_0)$$

$$vf_0 - wvf_1 = \frac{eE}{m}\frac{d\Phi_0}{dv} + \nu v\Phi_1$$

We have used here the expression for the collision integral of the asymmetric part of the distribution function $I_c[v_x\Phi_1(v)] = -\nu v_x\Phi_1(v)$.

The second term in the second equation is small compared with the first term, since $wf_1 = 3mf_0/M \ll f_0$ and we shall ignore it. According to eq. (2.21), the collision integral of the spherically symmetric part of the distribution function is

$$I_c(\Phi_0) = \frac{m}{M}\frac{\nu}{v^2}\frac{d}{dv}(v^3\Phi_0)$$

Using the above relations, we can transform the system of equations into

$$\left(\frac{mv^2}{Mw} - w\right) f_0 = \frac{eE}{3m\nu^2}\frac{d}{dv}(v^3\Phi_1) - \frac{m\nu}{Mv^2}\frac{d}{dv}(v^3\Phi_0)$$

$$vf_0 = \frac{eE}{m}\frac{d\Phi_0}{dv} + \nu v\Phi_1$$

Define a new variable $x = \frac{v}{w}\sqrt{\frac{m}{M}}$ and express the sought functions as

$$\Phi_0 = \frac{Mw}{m\nu} f_0 F_0, \quad \Phi_1 = \frac{f_0 F_1}{\nu}$$

Using the new notation, we have $f_0 = A \exp(-3x^2/2)$, so that $df_0/dx = -3xf_0$. Then we obtain the following system of equations for the functions F_0 and F_1:

$$(x^2-1)\exp\left(-\frac{2}{3}x^2\right)=\frac{1}{3x^2}\frac{d}{dx}\left[x^3\exp\left(-\frac{3}{2}x^2\right)(F_1-F_0)\right]$$

$$x\exp\left(-\frac{3}{2}x^2\right)=\frac{d}{dx}\left[F_0\exp\left(-\frac{3}{2}x^2\right)\right]+x\exp\left(-\frac{3}{2}x^2\right)F_1$$

The second equation yields $F_1 = 1 + 3F_0 - \frac{1}{x}\frac{dF_0}{dx}$; substituting this expression into the first equation, we obtain

$$2x^2-2-x\frac{dF_0}{dx}+\frac{2}{3x}\frac{dF_0}{dx}+\frac{1}{3}\frac{d^2F_0}{dx^2}=0$$

The solution of this equation is $F_0 = C + x^2$. The constant C can be found from the condition (4.26a) which now has the form $\int_0^\infty F_0(x)x^2 \times \exp(-3x^2/2)\,dx = 0$. We obtain $C = -1$ and

$$F_0 = x^2 - 1, \quad F_1 = 3x^2 - 4$$

Finally, eq. (4.28) for the longitudinal diffusion coefficient of electrons yields

$$\mathscr{D}_{||}=\frac{1}{N_i}\int(v_x-w)v_x\Phi_1(v)\,d\mathbf{v}=\frac{1}{3}\left\langle\frac{v^2}{\nu}F_1(v)\right\rangle$$

that is,

$$\mathscr{D}_{||}=\frac{Mw^2}{3m\nu}\equiv\frac{\langle v^2\rangle}{3\nu} \tag{4.37}$$

Under the conditions of this problem, when the collision frequency is independent of the collision velocity, the coefficients of transverse and longitudinal diffusion of electrons in the gas are identical.

§ 4.3. FORMATION OF CHARGED PARTICLES IN A WEAKLY IONIZED PLASMA

The equilibrium number density of charged particles in a weakly ionized plasma is maintained by competition of two processes — ionization of the gas atoms and disappearance of charged particles owing to their recombination or removal from the plasma. In this section we shall consider the production of charged particles in collisions between electrons and atoms. The variation of the number density of charged particles owing to ionization of atoms is described by the following equation:

$$\frac{dN_e}{dt} = k_{\mathrm{ion}} N_e N_a$$

Here N_e and N_a are the number densities of electrons and atoms, and k_{ion} is the ionization rate constant. The parameter $\nu_{\mathrm{ion}} = N_a k_{\mathrm{ion}}$ is known as the *frequency of ionization.*

Ionization is called direct if it occurs when an electron collides with an atom in the ground state. If free electrons are produced mainly in repeated collisions with electron, that is, by ionization of excited atoms, ionization is called *stepwise.* For direct ionization of atoms by an electron impact the ionization rate constant is

$$k_{\mathrm{ion}} = \langle v \sigma_{\mathrm{ion}} \rangle$$

where v is the electron velocity, σ_{ion} is the ionization cross section, and angle brackets denote the averaging over electron velocities. We shall discuss both ionization mechanisms.

Problem 4.29. Find the cross section of direct ionization of atoms by an electron impact. Assume that during collision the interaction between an incident electron and a valence electron is much stronger than their interaction with the atomic core, and that the valence electron is resting.

Under the assumptions of the problem the cross section of ionization coincides with the cross section of collision between the incident electron and the resting valence electron, when the energy transferred to the valence electron is higher than the ionization potential for the atom. According to eq. (A1.6), the cross section of collision between the electron with the energy E and the resting electron in which the energy in the range from $\Delta\varepsilon$ to $\Delta\varepsilon + d(\Delta\varepsilon)$ is exchanged is

$$d\sigma = \frac{\pi e^4 d(\Delta\varepsilon)}{E \Delta\varepsilon^2}$$

Integration of this equation from the ionization potential J to E yields for the ionization cross section

$$\sigma_{\mathrm{ion}} = \frac{\pi e^4}{E}\left(\frac{1}{J} - \frac{1}{E}\right) \tag{4.38a}$$

Equation (4.38a) is valid for one valence electron and is known as the *Thomson formula*. If ionization involves several valence electrons, summation should be performed.

> **Problem 4.30.** Calculate the cross section of direct ionization of atoms by an electron impact. Assume that the scattering of an incident electron by a valence electron is described by classical equations and the velocity distribution of valence electrons is spherically symmetric and determined by the motion of electrons in the Coulomb field of the atomic core. Analyze the case when the energy of the incident electron is much higher than the mean energy of valence electrons.

The conditions of the problem correspond mainly to small scattering angles of the incident electrons and, as discussed in A1.4, the variation of the momenta of the incident and valence electrons with scattering is

$$\Delta p = 2e^2/v\rho$$

where ρ is the impact parameter, and v is the relative collision velocity. Hence, the ionization cross section is

$$\sigma_{\text{ion}} = \int 2\pi\rho\, d\rho = \frac{8\pi e^4}{v^2}\left\langle \int \frac{d(\Delta p)}{\Delta p^3} \right\rangle$$

where we integrate over the exchanged energies exceeding the ionization potential and the angle brackets denote the averaging over the velocities of the valence electrons.

The variation of the energy of the valence electron in collision with the incident electron is

$$\Delta\varepsilon = \frac{(m\mathbf{u} + \Delta\mathbf{p})^2 - m^2u^2}{2m}$$

where $\mathbf{u}$ is the initial velocity of the valence electron. Hence, the transferred momentum is

$$\Delta p = \sqrt{(m\mathbf{u}\mathbf{n})^2 + 2m\Delta\varepsilon} - m\mathbf{u}\mathbf{n}$$

where $\mathbf{n}$ is the unit vector along the vector $\Delta\mathbf{p}$. Substituting this expression into the above equation for the ionization cross section, we obtain

$$\sigma_{\text{ion}} = \frac{8\pi e^4}{v^2}\int \left\langle \frac{m d(\Delta\varepsilon)}{[\sqrt{(m\mathbf{u}\mathbf{n})^2 + 2m\,\Delta\varepsilon} - m\mathbf{u}\mathbf{n}]^3 \sqrt{(m\mathbf{u}\mathbf{n})^2 + 2m\,\Delta\varepsilon}} \right\rangle$$

After averaging this expression over the angle between the vector **u** and **n**, we have

$$\sigma_{\text{ion}} = \frac{\pi e^4}{E}\left\langle \int_J^E \frac{d(\Delta\varepsilon)}{\Delta\varepsilon^3}\left(\Delta\varepsilon + \frac{2}{3}mu^2\right)\right\rangle$$

$$= \frac{\pi e^4}{E}\left[\frac{1}{J} - \frac{1}{E} + \frac{m\overline{u}^2}{3} + \left(\frac{1}{J^2} - \frac{1}{E^2}\right)\right]$$

If we take the velocity of the valence electron to be zero, this expression reduces to the Thomson formula. We assume that the valence electron is in the Coulomb field; then the mean kinetic energy of the electron is $m\overline{u}^2/2 = J$ and the above result can be rewritten as

$$\sigma_{\text{ion}} = \frac{\pi e^4}{E}\left(\frac{5}{3J} - \frac{1}{E} - \frac{2J}{3E^2}\right) \tag{4.38b}$$

When we consider this equation for low energies of the incident electron comparable with the ionization potential, we find a threshold for $E = J$ as in the Thomson formula. The limits of integration for both formulas were chosen in such a way that the ionization cross section is zero for $E = J$. According to eq. (4.38b), the maximum ionization cross section is $\sigma_{\max} \approx \pi e^4/2J^2$ for $E = 1.85J$, while the Thomson formula yields $\sigma_{\max} \approx \pi e^4/4J^2$ for $E = 2J$. If the energy of the incident electron is high compared with the ionization potential, the ionization cross section given by eq. (4.38b) is 5/3 times that given by the Thomson formula. Thus, the motion of the valence electron affects the magnitude of the cross section of ionization by an electron impact.

Problem 4.31. Compare the frequencies of ionization for atoms in the ground and excited states using the Thomson formula (4.38a). The velocity distribution for electrons has the Maxwellian form and the thermal energy of electrons is much higher than the ionization potentials I and I^* for atoms in the ground and excited states, respectively.

The Thomson formula (4.38 a) gives the ionization cross section $\sigma = \dfrac{\pi e^4}{E}\left(\dfrac{1}{I} - \dfrac{1}{E}\right)$. Therefore, the frequency of ionization (the frequency of production of one ion by one test electron) under the conditions of the problem is

$$\nu_{\text{ion}} = \frac{\pi e^4}{I^2}\sqrt{\frac{8T}{\pi m}}\exp(-I/T)N_0$$

where N_0 is the number density of atoms in the ground state. Hence, the ratio between the frequencies of ionization of atoms in the ground and excited states is

$$\frac{\nu_0}{\nu_{exc}} = \frac{N_0 \exp(-I/T)}{N_{exc} \exp(-I^*/T)} \left(\frac{I^*}{I}\right)^2 \tag{4.39}$$

where N_{exc} is the number density of atoms in the excited state.

Equation (4.39) shows that at thermodynamic equilibrium, when the number density of atoms in the excited state is given by the Boltzmann distribution, the production of free electrons is due mainly to ionization of excited atoms. The use of the Thomson formula was not essential for the derivation of eq. (4.39), since this formula gives a correct estimate of the ionization cross section. Equation (4.39) was derived for the Maxwellian energy distribution of electrons: when it is violated, the decrease of the energy distribution at high energies is sharper than that of the Maxwellian function. Since the fast electrons of the distribution tail are responsible for ionization, violations of the Maxwellian distribution form contribute to better conditions for stepwise ionization.

Other arguments can be put forward in favour of the stepwise mechanism of ionization in the case when the excited atoms are at thermodynamic equilibrium with electrons. In a system consisting of ions, electrons and atoms the inverse process for the ionization is the three-body recombination of electrons and ions in collisions with free electrons. The three-body recombination at first produces an excited atom with the energy of the order of the electron temperature, and after subsequent collisions with electrons the atom goes over to the ground state. Since at thermodynamic equilibrium the inverse process for recombination proceeds via the same mechanisms but in the opposite direction, the free electrons are produced mainly via ionization of excited atoms.

Problem 4.32. Determine the conditions of stepwise ionization of atoms in a positive gas-discharge column.

If ionization of excited atoms has a higher rate than direct ionization of atoms in the ground state, the overall process is termed stepwise ionization. Let us find the number density of atoms in a given excited state. Let τ^* be the effective lifetime of excited atoms determined both by the removal of the atoms to the walls and de-excitation radiation which freely leaves the gas-discharge column. The balance equation for the number density of excited atoms is

$$\frac{dN_{exc}}{dt} = 0 = N_e k_{0exc} N_0 - N_e k_{exc0} N_{exc} - \frac{N_{exc}}{\tau^*}$$

where k_{0exc} is the rate constant for the electron-impact excitation of atoms from the ground state to the given excited state, k_{exc0} is the rate constant for the inverse process, and N_e, N_0 and N_{exc} are the number densities of electrons and atoms in the ground state and the given excited state, respectively. If $\tau^* \to \infty$, the excited atoms are at thermodynamic equilibrium with electrons and their number density is given by the Boltzmann distribution $N_{exc} = \frac{g_{exc}}{g_0} \exp(-\Delta E/T)$, where g_0 and g_{exc} are the statistical weights of the ground and excited atomic states, and $\Delta E = I - I^*$ is the excitation energy. The relationship between the rate constants of excitation and de-excitation of atoms by the electron impact given by the balance equation is $k_{0exc} = k_{exc0} \frac{g_{exc}}{g_0} \exp(-\Delta E/T)$; it is still valid when thermodynamic equilibrium is violated. Then the balance equation for the number density of the excited atoms yields

$$N_{exc} = N_0 \frac{g_0}{g_{exc}} \exp(-\Delta E/T) \frac{1}{1 + N_e k_{exc0} \tau^*}$$

Substituting this expression into eq. (4.39), we obtain the condition for stepwise ionization

$$N_e k_{exc0} \tau^* \frac{g_{exc}}{g_0} \gg \left(\frac{I^*}{I}\right)^2 \tag{4.40}$$

If the condition (4.40) is satisfied for one of the excited atomic states, the stepwise ionization occurs in the positive gas-discharge column.

Problem 4.33. Find the relationship between the rate constants of stepwise ionization in two different gases with identical electron densities and temperatures. The electron velocity distributions have the Maxwellian form, and the electron temperature is much lower than the excitation energy for atoms.

The number density of free electrons produced by stepwise ionization per unit time is

$$\frac{dN_e}{dt} = N_e \sum_n N_n k_n$$

where N_e is the number density of electrons, N_n is the number density of atoms in the nth state, and k_n is the rate constant of direct ionization of these atoms. The rate constant of stepwise ionization k_{ion} is given by the equation $dN_e/dt = k_{ion} N_0$, where N_0 is the number density of atoms in the ground state. Hence, we have

$$k_{ion} = \sum_n k_n \frac{N_n}{N_0} \tag{4.41}$$

To derive the relationship between the rate constants of stepwise ionization in different gases, we shall take into account the fact that the properties of the highly excited atoms are independent of the atomic species since they are determined by the Coulomb interaction between electrons and atomic core charges. Therefore, the rate constants k_n of direct ionization of excited atoms are identical for atoms of different species. Moreover, the ratio $N_n^{(1)}/N_n^{(2)}$ between the number densities of two atomic species in the same highly excited state with the same number densities and temperatures of electrons is independent of the number of the state. Since at low electron temperatures the rate constant k_{ion} is determined by the ionization of highly excited atoms, we obtain .

$$\frac{k_{\text{ion}}^{(1)}}{k_{\text{ion}}^{(2)}} = s\,\frac{N_0^{(2)}}{N_0^{(1)}}$$

where $s = N_n^{(1)}/N_n^{(2)}$ and the superscripts (1) and (2) denote the atoms of the first and second species.

To find the factor s which is independent of the electron number density, let us analyze the case when the electron number density is high and the frequency of collisions between electrons and excited atoms is much higher than the frequency of de-excitation radiation of excited atoms. The number density of excited atoms is given by the Boltzmann distribution:

$$N_n = \frac{g_n}{g_0} N_0 \exp\left[-(I - I_n)/T_e\right]$$

Here T_e is the electron temperature, g_0 and g_n are the statistical weights of the respective states, and I and I_n are the ionization potentials for atoms in the ground and excited states. Then we obtain

$$s = \frac{g_{2a}g_{1i}}{g_{1a}g_{2i}}\,\frac{N_0^{(1)}}{N_0^{(2)}} \exp\left[(I_2 - I_1)/T_e\right]$$

where g_{1a}, g_{2a} and I_1, I_2 are the statistical weights and ionization potentials for the respective atoms in the ground state, and g_{1i} and g_{2i} are the statistical weights of the ions of the respective atom. Here we used the fact that the statistical weight of a highly excited atom is $g_n = g_e g_i n^2$, where g_i, $g_e=2$ are the statistical weights of the ion and electron, respectively, and n is the principal quantum number of the bound electron.

Thus, we obtain the following relationship between the rate constants of stepwise ionization for two different atomic gases with the identical number densities and temperatures of electrons:

$$k_{\text{ion}}^{(1)} \frac{g_{1a}}{g_{1i}} \exp(I_1/T_e) = k_{\text{ion}}^{(2)} \frac{g_{2a}}{g_{2i}} \exp(I_2/T_e) \tag{4.42}$$

Problem 4.34. Find the dependence of the ionization rate constant on the electron temperature for a high number density of electrons.

If the electron number density is high, de-excitation of atoms is insignificant and the number density of excited atoms is given by the Boltzmann distribution $N_n = \frac{g_n}{g_0} N_0 \exp(-\varepsilon_n/T_e)$, where ε_n is the excitation energy of the nth state. The ionization frequency is

$$\nu_{\text{ion}} = k_{\text{ion}} N_0 = \sum_n k_n N_n$$

Since inelastic collisions between electrons and atoms result mainly in the energy transfer of the order of the electron temperature, the main contribution to the above sum is made by the excited atoms with the ionization energy of the order of the temperature. Since $g_n = g_i g_e n^2$, where $g_i, g_e = 2$ are the statistical weights of the ion and electron and n is the principal quantum number of the excited atoms the direct ionization of which yields free electrons, we obtain

$$k_{\text{ion}} \sim \frac{g_i}{g_0} n^3 \langle v\sigma \rangle \exp(-I/T) \sim \left(\frac{me^4}{\hbar^2 T_e} \right)^{3/2} \langle v\sigma \rangle \exp(-I/T_e) \frac{g_i}{g_0}$$

where $v \sim \sqrt{T_e/m}$ is the electron velocity, and the ionization cross section σ for the excited atom with the ionization energy $me^4/\hbar^2 n^2 \sim T_e$ is of the order of magnitude of the cross section for transfer of energy of the order of T_e in collision between two electrons with energies of the order of T_e ($\sigma \sim e^4/T_e^2$). Hence, we have

$$k_{\text{ion}} = A \frac{g_i}{g_0} \frac{me^{10}}{\hbar^3 T_e^3} \exp(-I/T_e) \tag{4.43}$$

where the proportionality factor A is of the order of unity.

Problem 4.35. Find the rate constant for stepwise ionization of atoms, using the principle of detailed balancing and the equation for the recombination rate constant at low temperatures and high electron number densities.

We consider thermodynamic equilibrium in a plasma with a high electron number density, when the processes of de-excitation of the excited atoms are insignificant. The equilibrium condition shows that the number of free electrons produced by stepwise ionization per unit time is equal to the number of electrons disappearing per unit time owing to three-body recombination ($e + e + A^+ \rightarrow e + A$), that is, the process which is inverse with respect to stepwise ionization. This condition yields

$$N_e N_a k_{ion} = N_e N_i \alpha$$

where N_a, N_e, and N_i are the number densities of atoms, electrons and ions, respectively, k_{ion} is the rate constant of stepwise recombination, and α is the rate constant of three-body recombination. Since recombination occurs with three-body collisions, the rate constant of recombination should be proportional to the number density of electrons: $\alpha = N_e \varkappa$. Now we obtain

$$k_{ion} = \frac{N_e N_i}{N_a} \varkappa$$

Since the plasma is at thermodynamic equilibrium, we can employ the Saha equation (2.14) relating the number densities of electrons, ions and atoms to each other:

$$\frac{N_e N_i}{N_a} = \frac{g_e g_i}{g_a} \left(\frac{m T_e}{2\pi\hbar^2} \right)^{3/2} \exp(-I/T_e)$$

Here I is the ionization potential of the atom, T_e is the electron temperature, and g_e, g_i and g_a are the statistical weights of the electron, ion and atom, respectively. Hence,

$$k_{ion} = \frac{g_e g_i}{g_a} \left(\frac{m T_e}{2\pi\hbar^2} \right)^{3/2} \exp(-I/T_e) \varkappa(T_e)$$

Now let us make use of the expression for the recombination rate constant in the case of high number density and low temperature of electrons (see Problem 4.41):

$$\varkappa = \frac{B e^{10}}{\sqrt{m}\, T_e^{9/2}}$$

Here the factor B is of the order of unity. Then the rate constant of stepwise ionization is

$$k_{ion} = A \frac{g_i}{g_a} \frac{m e^{10}}{\hbar^3 T_e^3} \exp(-I/T_e), \quad A = \frac{2B}{(2\pi)^{3/2}} \quad (g_e = 2) \tag{4.44}$$

Problem 4.36. Compare the rate constants for direct and stepwise ionization for high number density and low temperature of electrons.

If the electron number density is high, so that de-excitation of excited atoms can be ignored and the electron temperature is low, the rate constant of stepwise ionization is given by eq. (4.43):

$$k_s = A \frac{g_i}{g_0} \frac{m e^{10}}{\hbar^3 T_e^3} \exp(-I/T_e)$$

Here T_e is the electron temperature, I is the ionization potential of the atom, and the constant A is of the order of unity.

In the vicinity of the ionization threshold the cross section of ionization by an electron impact varies linearly with energy:

$$\sigma_{\text{ion}} = \sigma_0(\varepsilon/I - 1), \qquad \varepsilon > I$$

Here ε is the energy of the incident electron, and σ_0 is of the order of the atomic cross section. If the electron velocity distribution has the Maxwellian form,

$$f(\varepsilon) = \frac{2\sqrt{\varepsilon}}{\pi^{1/2}T_e^{3/2}} \exp(-\varepsilon/T_e), \qquad \int_0^\infty f\,d\varepsilon = 1$$

we obtain the following expression for the rate constant of direct ionization:

$$k_d = \int_I^\infty f(\varepsilon)\sqrt{\frac{2\varepsilon}{m}}\,\sigma_{\text{ion}}\,d\varepsilon = \sqrt{\frac{8T_e}{\pi m}}\,\sigma_0 \exp(-I/T_e), \qquad T_e/I \ll 1$$

Thus, we have

$$\frac{k_d}{k_s} = \frac{g_0\sigma_0}{Ag_i a_0^2}\sqrt{\frac{8}{\pi}}\left(\frac{\hbar^2 T_e}{me^{10}}\right)^{7/2} \ll 1$$

Here we have used the fact that $T_e \ll I \sim \dfrac{me^2}{\hbar^2}$.

Thus, in the infinite system with high number density and low temperature of electrons free electrons are produced via stepwise ionization of atoms.

§ 4.4. RECOMBINATION OF CHARGED PARTICLES IN PLASMA

Recombination produces the variation of the number densities N_+ and N_- of the positive ions and negatively charged particles (electrons or negative ions) described by the equation

$$\frac{dN_+}{dt} = \frac{dN_-}{dt} = -\alpha N_+ N_- \tag{4.45}$$

where α is the recombination coefficient. Recombination of charged particles results in a release of energy equal to the binding energy of the recombining particles. If this energy is absorbed by the internal degrees of freedom of the particles, then recombination is a two-body process and the recombination coefficient is independent of the particle density. If the excess energy is transferred to a third particle (three-body recombination), the recombination coefficient is proportional to the number density of the particles of the third species.

Problem 4.37. Determine the coefficient of three-body recombination of a positive and negative particles as a function of the parameters describing their interaction with a third particle.

Assume that the charged particles are in thermodynamic equilibrium with the third species, so that their kinetic energy is of the order of the system's temperature T. Then recombination of these particles necessitates their approaching each other to a distance for which the potential of interaction of the charged particles is of the order of the thermal energy. Indeed, the third particle acquires energy from one of the charged particles and the transferred energy is of the order of the kinetic energy of the charged particle. For recombination of the charged particles to occur, the transferred energy should be higher than the kinetic energy of the charged particles before the collision.

The separation b of the recombining particles for which the potential of interaction between them is of the order of their thermal energy $[U(b) \sim T]$ will be called below the critical radius. The frequency of disappearance of positive ions is the product of the frequency $N_3\bar{v}\sigma$ of collisions of one charged species with the particles of the third species (the energy transferred to the particle of the third species in this collision is of the order of T) and the probability N_-b^3 that at this moment a negatively charged particle is in the critical region. Here N_3 is the number density of the third species, $\bar{v}$ is the relative velocity of collision between the particle of the third species and one of the charged particles, and σ is the cross section of the collision in which one of the charged particles transfers the energy of the order of T to the particle of the third species. Hence, the coefficient of three-body recombination is

$$\alpha \sim N_3\bar{v}\sigma b^3 \tag{4.46}$$

Here we assumed that simultaneous collision of a charged particle with two particles of the third species is unlikely, so that the mean free path of the charged particle with respect to collisions with the particles of the third species is much greater than the size of the critical region:

$$N_3\sigma b \ll 1 \tag{4.47}$$

This approach to the analysis of three-body recombination is known as the *Thomson theory*. This theory is applicable if the reduced mass of the charged particles is of the order of the mass of the third particle. In each of the strong collisions with the third particle the transferred energy is of the order of T, so that the cross section σ in eq. (4.46) is of the order of the cross section of the elastic collision of the third particle with the charged particles.

The Thomson theory estimates the recombination coefficient in the case when the bound state of two charged particles is produced after a single collision of these particles with a third particle. The theory is based on classical mechanics, that is, it assumes that the spacing of the levels of the bound system of the two charged particles is small. Therefore, the theory is not applicable, for instance, to the formation of negative ions in three-body collisions of electrons with gas atoms (the electron and the atom have only one bound state).

The Thomson theory is very useful since it provides a simple method for estimating the recombination coefficient and determining its dependence on plasma parameters. This theory has a fairly simple physical meaning and only one constant in it remains indefinite, while accurate calculations of the recombination coefficient for each specific recombination mechanism present a cumbersome mathematical problem.

Problem 4.38. Determine the dependence of the coefficient of electron-ion three-body recombination, where the third body is an electron, on the plasma parameters.

Recombination of this type is described by the equation

$$e + e + A^+ \to e + A$$

The critical radius b is found from the relation $e^2/b \sim T$, the collision velocity of electrons is $\bar{v} \sim \sqrt{T/m}$ and for the Coulomb cross section of energy transfer $\sigma \sim e^4/T^2$ we find from eq. (4.46) the recombination coefficient

$$\alpha = a\sqrt{\frac{T}{m}}\left(\frac{e^2}{T}\right)^5 N_e \tag{4.48}$$

where the factor a is of the order of unity, and N_e is the electron number density. Here the condition of applicability of the Thomson theory coincides with the condition of plasma ideality:

$$\frac{e^6 N_e}{T^3} \ll 1$$

Problem 4.39. Determine the dependence of ion-ion three-body recombination coefficient, where the third body is an atom, on the plasma parameters.

Recombination is described by the equation

$$A^- + B^+ + C \to A + B + C$$

Here the cross section of the collision between the neutral atom C and an ion corresponds to polarization interaction and is of the order of $\sigma_{el} \sim \sqrt{\beta e^2/T}$, where β is the polarizability of the atom C. The recombination coefficient is

$$\alpha \sim \frac{e^6 N_a}{T^3}\bar{v}\sigma_{el} \tag{4.49}$$

where N_a is the number density of the atoms C, and $\bar{v}$ is the relative velocity of the ion-atom collision. We assume that the ion mass is of the order of the atomic mass.

Expression (4.49) is valid if the mean free path of ions with respect to collisions with atoms $\lambda \sim 1/N_a\sigma_{el}$ is much smaller than the size $b \sim e^2/T$ of the critical region. Thus, the condition of applicability of eq. (4.49) is

$$N_a \frac{e^2}{T} \sigma_{el} \ll 1 \tag{4.50}$$

Problem 4.40. Find the rate constant of formation of the molecular ions from the atomic ions in three-body collisions with gas atoms as a function of the plasma parameters (temperature and density of plasma, particle masses, and atom polarizability).

The process is described by the equation

$$A^+ + B + C \rightarrow AB^+ + C$$

Though this reaction does not involve recombination of charged particles, it has a similar character and the rate constant for it, defined by the equation

$$\frac{dN_{A^+}}{dt} = -kN_{A^+}N_B$$

can be calculated in the same way as it was done for the recombination coefficient. We shall assume that the masses of the particles A, B and C are of the same order of magnitude. Then the critical radius b is given by $\beta_B e^2/b^4 \sim T$, where β_B is the polarizability of the atom B, and the distance ρ, between the particles A^+ and C for which the energy transferred to the particle C is of the order of $T (\sigma \sim \rho^2)$ corresponds to the impact parameter of capture and can be found from the relation $\beta_C e^2/\rho^4 \sim T$, where β_C is the polarizability of the atom C. Now eq. (4.46) yields the following expression for the rate constant:

$$k = aN_C \left(\frac{\beta_B e^2}{T}\right)^{3/4} \left(\frac{\beta_C e^2}{M}\right)^{1/2} \tag{4.51}$$

where the dimensionless factor a is of the order of unity, and M is the particle's mass.

Problem 4.41. Employing dimensional analysis, find the rate constant of the process $A + 2B \rightarrow AB + B$ as a function of the parameters of the problem. The temperature T of a plasma is much lower than the bonding energy for the ground state of the system AB, the mass m of the particle B is much lower or of the order of the mass of the particle A, and the relationship $U(R) = -C/R^n$ describes the potential of interaction between the recombining particles in the range of separation distances R, in which the potential is of the order of the thermal energy.

According to the Thomson theory (see Problem 4.37), three-body recombination gives rise mostly to the bound states of the recombining particles with the energy of the order of the plasma temperature. Therefore, the rate constant of the process depends only on the parameters T, m and C. We assume that the interaction between the particles forming the bound state and the interaction of one of them with the third particle acquiring the excess energy have the same character. Since the rate of the process is determined by the reduced mass for each of the pairs of interacting particles, the result is independent of the mass of the particle A if it is much greater than the mass m of the particle B.

The constant rate of the three-body process is defined by the equation $\nu = \varkappa N_A N_B^2$, where ν is the frequency of the three-body recombination, and N_A and N_B are the number densities of the particles A and B. With this definition $\varkappa$ is independent of the number densities of the particles and is expressed in terms of the parameters T, m and C. There is only one combination of these parameters with the required dimension (cm^6/s):

$$\varkappa = a \frac{(|C|)^{5/n}}{T^{5/n-1/2}\sqrt{m}}$$

Here a is a dimensionless factor of the order of unity.

For instance, for the process $2e + A^+ \rightarrow A + e$ for which $|C| = e^2$, where e is the electron's charge, the above expression yields

$$\varkappa = a \frac{e^{10}}{T^{9/2} m^{1/2}}$$

This expression coincides with eq. (4.48). For the process $A^+ + 2B \rightarrow AB^+ + B$ for which $|C| = \beta e^2$, where β is the polarizability of the atom B, we obtain the expression

$$\varkappa = a \frac{(\beta e^2)^{5/4}}{T^{3/4} m^{1/2}}$$

which is identical to eq. (4.51) if the particles B and C are identical.

Problem 4.42. Express the rate constant for the process $A + B + C \rightarrow AB + C$ in terms of the rate constant for the inverse process $AB + C \rightarrow A + B + C$. Here A, B and C are atoms or atomic ions.

The balance equation for the number densities of the particles is

$$\frac{dN_A}{dt} = \sum_{v,I} N_{AB} N_C k_d(v, I) - \varkappa N_A N_B N_C$$

where N_A, N_B and N_C are the number densities of the particles A, B and C, N_{AB} is the number density of the molecules AB which are in the vibrational state v and the rotational state I, $k_d(v, I)$ is the rate constant for decomposition of these molecules, and $\varkappa$ is the rate constant for the three-body process.

If the system is at thermodynamic equilibrium, the relationship between the number densities of atoms and molecules in the given vibrational-rotational state is given by the Saha equation (2.14)

$$\frac{N_A N_B}{N_{AB}} = \frac{g_1 g_2}{(2I+1) g_m} \left(\frac{\mu T}{2\pi\hbar^2}\right)^{3/2} \exp(-D^*/T)$$

where g_1, g_2 and g_m are the statistical weights of the atoms A and B and the molecule AB, determined by their electronic state, μ is the reduced mass of the molecule AB, and D^* is the energy of dissociation of the molecule AB in the given state. To maintain thermodynamic equilibrium in the system, the following relationship between the rate constants of the direct and inverse processes should be satisfied:

$$\varkappa = \frac{g_m}{g_1 g_2} \left(\frac{2\pi\hbar^2}{\mu T}\right)^{3/2} \sum_{v,I} (2I+1) k_d (v, I) \exp(D^*/T)$$

Naturally, this relationship between the rate constants is conserved if thermodynamic equilibrium is maintained only between the atoms.

Problem 4.43. Calculate the rate constant of formation of the molecular ions in the process $A^+ + 2A \rightarrow A_2^+ + A$, using the results of Problem 4.42. Make use of the impulse approximation to calculate the rate constant of decomposition of the molecular ion.

According to the results of Problem 4.42, we can express the rate constant of formation of the molecular ions in terms of the rate constant of decomposition of the ions which is given by

$$k_d = k_c W_d$$

where $k_c = 2\pi\sqrt{3\beta e^2/2M}$ is the rate constant of capture of the atom by the molecular ion, M is the mass of the atom A, $2M/3$ is the reduced mass of the atom and the molecular ion, W_d is the probability that the three-body system decomposes into three particles, and β is the polarizability of the atom A. This equation describes the physical picture of the process in which only strong interaction between the particles can lead to decomposition of the system and this interaction is possible only when capture results in a decrease in the distance between the particles. The main difficulty is to find W_d which is determined by the motion of the three particles. We shall use for this a considerably simplified model. It will be seen that each of the assumptions of the model leads to overestimation of results which, thus, should be regarded as the upper limits of the sought rate constant.

We shall assume that the levels of the molecular ion in the bound system are not close to the ground state, so that the classical equations of the motion of particles can be employed. Moreover, we shall ignore the interaction between the atoms, which is small compared with the polarization interaction between the ion and the atom in the range of separation distances which is of interest for us. Apart from these justified assumptions we shall make other assumptions. One assumption is that of impulse approximation according to which the time of collision between the ion and the atom is so short that the distance between the ion and the bound atom does not change during the collision. Another assumption is that momentum exchange between the ion and the free atom occurs as in the head-on collision. Then, if before the collision the velocity of the free atom is $\mathbf{v}_1$, the velocity of the ion is $\mathbf{v}_2$, and the velocity of the bound atom is $\mathbf{v}_3$, after the collision these velocities are $\mathbf{v}_2$, $\mathbf{v}_1$ and $\mathbf{v}_3$, respectively.

Denote the energy of dissociation of the molecular ion in the given state by $\mathscr{E}$, the magnitude of the potential of interaction between the ion and the atom in the molecular ion for a given distance between them by U, and the velocity of the molecular ion by $\mathbf{V} = (\mathbf{v}_2 + \mathbf{v}_3)/2$. It is convenient to express the velocities $\mathbf{v}_2$ and $\mathbf{v}_3$ in terms of $\mathbf{V}$ and the relative velocity $v = \sqrt{2(U - \mathscr{E})/\mu}$ of the particles in the molecular ion before the collision (here $\mu = M/2$). Using the above assumptions, we find that the relative velocity of the particles in the molecular ion after the collision is

$$\mathbf{v}_r = \mathbf{v}/2 + \mathbf{V} - \mathbf{v}_3 = \mathbf{v}/2 + \mathbf{u}$$

where $\mathbf{u}$ is the relative velocity of the molecular ion and the atom before the collision. If $\mu v_r^2 \geqslant U(r)$, the collision results in decomposition of the molecular ion. Then the probability of decomposition is

$$W_d(r, \mathscr{E}) = \begin{cases} 0 \quad \text{for} \quad u < \sqrt{\dfrac{2U(r)}{\mu}} - \dfrac{v}{2} \\[2ex] \dfrac{\left(u + \dfrac{v}{2}\right)^2 - \dfrac{2U(r)}{\mu}}{2uv} \quad \text{for} \quad \sqrt{\dfrac{2U(r)}{\mu}} - \dfrac{v}{2} < u < \sqrt{\dfrac{2U(r)}{\mu}} + \dfrac{v}{2} \\[2ex] 1 \quad \text{for} \quad u > \sqrt{\dfrac{2U(r)}{\mu}} + \dfrac{v}{2} \end{cases}$$

We can simplify this equation writing it as

$$W_d(r, \mathscr{E}) = \begin{cases} 0 \quad \text{for} \quad u < \sqrt{\dfrac{2U(r)}{\mu}} \\[2ex] 1 \quad \text{for} \quad u > \sqrt{\dfrac{2U(r)}{\mu}} \end{cases}$$

According to the results of Problem 4.42, the rate constant of the three-body process is

$$\varkappa = g\left(\frac{4\pi\hbar^2}{MT}\right)^{3/2}\int\frac{d\mathbf{r}\,d\mathbf{p}}{(2\pi\hbar)^3}\,\delta\left(-\mathscr{E}+U-\frac{p^2}{2\mu}\right)d\mathscr{E}\,k_d\exp(\mathscr{E}/T)$$

where $g = g_m/g_1g_2$, $\frac{d\mathbf{r}\,d\mathbf{p}\,d\mathscr{E}}{(2\pi\hbar)^3}\,\delta\left(U-\mathscr{E}-\frac{p^2}{2\mu}\right)$ is the number of states in the given element of the phase space, and $p = \mu v$. Averaging k_d over the relative velocity of the molecular ion and the atom with the Maxwellian velocity distribution, we obtain

$$k_d = 2\pi\left(\frac{3\beta e^2}{2M}\right)^{1/2}\frac{2}{\sqrt{\pi}\,T^{3/2}}\int\limits_{4U/3}^{\infty}\exp(-E/T)E^{1/2}\,dE$$

The lower limit of integration can be explained as follows. Decomposition of the ion occurs when $\mu u^2/2 \equiv Mu^2/4 \geqslant U$. The relative energy of the collision between the molecular ion and the atom is $E = Mu^2/3$, since the reduced mass of these particles is $2M/3$. Hence, decomposition of the molecular ion occurs for the relative collision energy $E = 4U/3$. Now we obtain the following expression:

$$\varkappa = g\,\frac{16\sqrt{2\pi}\,(3\beta e^2)^{1/2}}{M^{1/2}T^3}\int\limits_{\mathscr{E}<U}\sqrt{U-\mathscr{E}}\,r^2\,dr\exp(\mathscr{E}/T)\,d\mathscr{E}\int\limits_{4U/3}^{\infty}\exp(-E/T)E^{1/2}\,dE$$

The main contribution to the integral is made by the range of r in which $U \sim T$, that is, the interaction has the polarization character ($U = \beta e^2/2r^4$). Taking into account this fact and using the new variables $x = U/T$, $y = \mathscr{E}/T$ and $z = E/T$, we obtain

$$\varkappa = g\,\frac{(\beta e^2)^{5/4}}{M^{1/2}T^{3/4}}\,\frac{4\pi\sqrt{3}}{\sqrt[4]{2}}\int\limits_0^{\infty}\frac{dx}{x^{7/4}}\int\limits_0^{x}\sqrt{x-y}\,dy\exp(y)$$

$$\times\int\limits_{4x/3}^{\infty}\exp(-z)\,z^{1/2}\,dz = 13.6g\,\frac{(\beta e^2)^{5/4}}{M^{1/2}T^{3/4}}$$

We see that the dependence of the process rate constant on the parameters βe^2, M and T coincides with those found in Problems 4.40 and 4.41.

Problem 4.44. A monochromatic electron beam for a short time passes through a gas. After that the ratio between the currents of the molecular and atomic ions is measured. Relate this ratio to the rate constants of the reactions in the gas which are described by the following equations

$$e + \mathrm{A} \xrightarrow{k_i} \mathrm{A}^+ + e, \qquad \mathrm{A}^* + \mathrm{A} \xrightarrow{k_r} \mathrm{A}_2^+ + e$$

$$e + \mathrm{A} \xrightarrow{k_e} \mathrm{A}^* + e, \qquad \mathrm{A}^* \xrightarrow{\frac{1}{\tau_d}} \mathrm{A} + \hbar\omega$$

Here A and A* are the atom and the excited atom from which the molecular ion is produced, and the symbols above the arrows denote the rate constants of the reaction time τ_d.

The balance equations for the number densities of particles are

$$\frac{dN_1}{dt} = N_e N_a k_i, \qquad \frac{dN^*}{dt} = N_e N_a k_e - N^*\left(\frac{1}{\tau_d} + N_a k_r\right)$$

$$\frac{dN_2}{dt} = N^* N_a k_r$$

where N_1 and N_2 are the number densities of the atomic and molecular ions, and N_a and N^* are the number densities of atoms in the ground and excited states. The solutions of these equations for $N_a k_r t_0 \ll 1$, where $t = t_0$ is the time of the electron pulse end, are

$$N_1(t_0) = N_a k_i \int_0^{t_0} N_e dt, \qquad N^*(t_0) = N_a k_e \int_0^{t_0} N_e\, dt, \qquad N_2 = 0$$

Solving the balance equations for $t > t_0$ when $N_e = 0$ and using the above initial conditions, we find the ratio of the currents of the molecular and atomic ions:

$$\frac{N_2}{N_1} = \frac{k_e}{k_i} \frac{1 - \exp\left[-\left(\frac{1}{\tau_d} + N_a k_r\right)\tau\right]}{1 + (N_a k_r \tau_d)^{-1}}$$

Here τ is the time starting from the end of the electron pulse. The above expression is the ratio between the numbers of the molecular and atomic ions formed by the moment τ. If the ions formed in the gas are instantaneously removed from it and the ratio between the numbers of the molecular and atomic ions can be measured at any moment, then experimental variation of the gas density can be used to find the parameters k_e/k_i, k_r and τ_d. If the ratio between the total numbers of the molecular and atomic ions can be measured for $t \to \infty$, this experimental technique can be used to find the parameters k_e/k_i and $k_r\tau_d$,

Problem 4.45. Compare the time of adjustment of the temperature and of the number density of particles in a plasma to its equilibrium value.

We assume here that the time of variation of the external conditions is short compared with the relaxation time for plasma. To describe the approach to equilibrium, we have to solve simultaneously the balance equations for the electron number density and the electron temperature. The balance equations for the electron number density are

$$\frac{dN_e}{dt} = -\alpha N_e^2 + \beta N_e N_a$$
$$\frac{dN_e}{dt} = -kN_e^3 + \beta N_e N_a \tag{4.52}$$

where the first equation is valid for binary recombination (or three-body recombination involving atoms) and the second equation for three-body recombination with electrons as the third body; α and kN_e are the recombination coefficients, β is the ionization coefficient, N_e is the electron number density, N_a is the number density of atoms ($N_e \ll N_a$) and the plasma is quasineutral ($N_e = N_i$). Assume that the electron temperature does not vary during relaxation of the electron number density. Then in the first equation $\beta = \alpha N_0/N_a$ and in the second equation $\beta = kN_0^2/N_e$, where N_0 is the equilibrium number density of electrons at the given temperature. Now the balance equations (4.52) reduce to

$$dN_e/dt = -\alpha(N_e - N_0)N_e$$
$$dN_e/dt = -k(N_e^2 - N_0^2)N_e$$

so that the time of relaxation of the electron number density is $\sim (\alpha N_0)^{-1}$ in the first case and $\sim (kN_0^2)^{-1}$ in the second case.

Let us compare the above frequencies of relaxation of the electron number density αN_0 and kN_0^2 with the frequency ν_{ea} of relaxation of the electron temperature. Assume that the electron equilibrium temperature is established owing to the elastic collisions between electrons and atoms. Then we have $\nu_{ea} \sim \dfrac{m}{M} N_a v \sigma_{el}$, where σ_{el} is the cross section of elastic collisions between electrons and atoms. If the recombination coefficient α corresponds to the recombination of electrons and molecular ions, we have $\alpha N_0 \sim v\sigma_r N_0$, where σ_r is the cross section of recombination of the electron and ion. Typically, $\sigma_r \lesssim \sigma_{el}$, so that when

$N_0 \ll \frac{m}{M} N_a$, the time of temperature relaxation is shorter than the time of number density relaxation.

We see that in the range of electron number densities for which the electron temperature can be reasonably defined the frequencies of relaxation of the electron number density and the electron temperature show no definite relationship. If the frequency of relaxation of the electron temperature is determined by collisions between electrons and gas particles (excitation of the electronic states or vibrational levels of the molecule), in which the exchanged energy is greater than in elastic collisions, then relaxation of temperature is faster. Relaxation of temperature is faster if recombination is due to three-body collisions of electrons and ions.

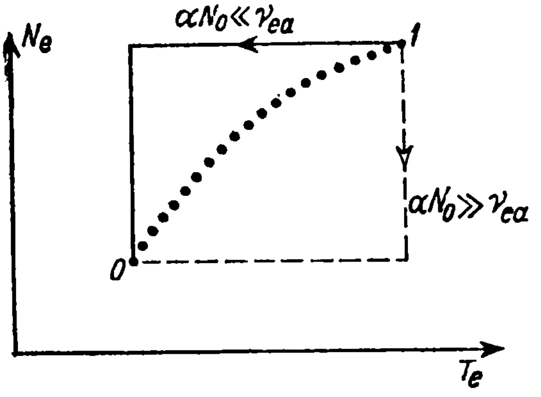

Fig. 4.4. Relaxation of electron number density and electron temperature in a plasma.

Figure 4.4 shows two limiting cases of variation of the state of a weakly ionized plasma described by the electron temperature and the electron number density. The initial state of the system is denoted by *1* and the final equilibrium state by *0*. The solid line corresponds to the case when the frequency of energy relaxation is much higher than the frequency of density relaxation, and the dash line corresponds to the inverse case. The dotted line corresponds to the Saha equation (2.14). In the second limiting case the electron number density at each moment has the equilibrium value corresponding to the given temperature. If the relaxation frequencies for temperature and number density are of the same order of magnitude, the line describing the approach to equilibrium lies between the lines corresponding to the limiting cases.

Problem 4.46. Derive the equation for finding the recombination coefficient if the frequencies of transitions between the states of recombining particles are known.

Denote by $W(n, n', t)$ the probability that a system of recombining particles, which was initially in the state n, goes over to the state n' during time t, and by $W(n, t)$ the probability that the system reaches the ground state by the time t. Then, by definition, we have

$$W(n, t) = 1 - \sum_{n'} W(n, n', t) \tag{4.53}$$

and the function $W(n, n', t)$ satisfies the Smoluchowski equation (1.66):

$$W(n, n', t+t') = \sum_k W(n, k, t)\, W(k, n', t')$$

In the limit $t' \to 0$ the Smoluchowski equation yields

$$\frac{\partial W(n, n', t)}{\partial t} = \sum_k w_{nk} W(k, n', t) - W(n, n', t) \sum_k w_{nk}$$

where $w_{nk} = \lim\limits_{t' \to 0} \dfrac{W(n, k, t')}{t'}$ is the probability of transition from the state n to the state k per unit time, and summation is performed over all the states apart from the ground state. The normalization condition $\sum\limits_k W(n, k, t') = 1$ was used here. Performing summation over the states n' in the above equation and using eq. (4.53), we obtain

$$\frac{\partial[1 - W(n, t)]}{\partial t} = \sum_k w_{nk}[1 - W(k, t)] - [1 - W(n, t)] \sum_k w_{nk} \qquad (4.54)$$

Denote by

$$\tau(n) = \int_0^\infty [1 - W(n, t)]\, dt = \int_0^\infty t\, \frac{\partial W(n, t)}{\partial t}\, dt$$

the mean time of reaching the ground state for the system of recombining particles which was initially in the state n. Since $W(n, 0) = 0$ and $W(n, \infty) = 1$, integration of eq. (4.54) with respect to time yields the following equation for $\tau(n)$:

$$-1 = \sum_k w_{nk}\tau(k) - \tau(n) \sum_k w_{nk} \qquad (4.55)$$

Let us relate the recombination coefficient α to the mean time τ in which a system of initially free recombining particles goes over to the ground state corresponding to the lowest bound state of the particles. The definition of the probability $W(n, t)$ that the system of recombining particles goes over to the ground state shows that this function satisfies the equation

$$W(n, t + t_1) = W(n, t) + [1 - W(n, t)] \sum_{n'} W(n, n', t)\, W(n', t_1)$$

which follows directly from the Smoluchowski equation. After differentiation of this equation with respect to t_1 and integration of the result over t we obtain in the limit $t_1 \to 0$:

$$1 = \int_0^\infty [1 - W(n, t)]\, dt \sum_{n'} W(n, n', t) \left.\frac{\partial W(n', t_1)}{\partial t_1}\right|_{t_1=0} \qquad (4.56)$$

Since the statistical weights of the continuous spectrum states are high, the time of recombination is much longer than the characteristic time of the transition between the states of the continuous spectrum of the recombining particles. For instance, in the case of electron-ion recombination the number of the available continuous-spectrum states is of the order of $(mT/\hbar^2)^{3/2}/N_e$, that is, it is very high for the ideal low-temperature plasma. Therefore, if the time t is greater than the time of the transition between the continuous spectrum states, $W(n, n', t)$ is independent of the initial state and coincides with the distribution $f(n')$ over the continuous spectrum states n'; for these states $\sum_{n'} f(n') = 1$. Generally speaking, we should take into account some discrete-spectrum states to which the system can go over in time t. However, owing to their low statistical weights, the contribution of the discrete-spectrum states to the sum over n' in eq. (4.56) is small and can be ignored. Now we can rewrite eq. (4.56) as

$$1 = \tau \sum_{n'} f(n') \left.\frac{\partial W(n', t_1)}{\partial t_1}\right|_{t_1=0}$$

According to the definition of the recombination coefficient α, we can write

$$\alpha N_i = \sum_{n'} f(n') \left.\frac{\partial W(n', t)}{\partial t}\right|_{t=0}$$

where N_i is the number density of ions and summation is performed over the continuous-spectrum states. Then we have

$$\alpha = (\tau N_i)^{-1} \tag{4.57}$$

Under the conditions of the problem (electron recombination in the ideal plasma, high statistical weight of the continuous-spectrum states in the general case) the dependence of the recombination time on the initial continuous-spectrum state is weak.

Thus, in the cases when recombination occurs via multiple transitions the recombination time can be found from eq. (4.55), and the simple equation (4.57) relates this time to the recombination coefficient.

Problem 4.47. Find the coefficient of the electron-ion recombination proceeding via the formation of an autoionized atomic state.

The autoionized state is the bound state of the electron and ion whose excitation energy is higher than the potential of ionization of the atom. Several electrons are excited in the autoionized state and if the excitation energy is transferred to one electron, the system decomposes and the electron escapes. The simplest example of the autoionized atomic state is the state of a helium atom with two electrons on excited levels. Owing to interaction between the electrons the system breaks down giving rise to a free electron and a helium ion in the ground state.

Another example is the bound system of an excited ion and an electron, in which the energy of the electron-ion bonding is lower than the ion excitation energy (for instance, the ion Xe$^+$ ($^2P_{\frac{1}{2}}$) and an electron). In contrast to the previous example, here the whole electron shell is excited and the interaction of the electron with it results in the breakdown of the system. It should be noted that owing to weak correlation between electrons the lifetime of autoionized states proves to be considerably longer than the characteristic atomic time ($\sim 10^{-17}$ s)*, so that such states are quasi-stationary.

The frequency of electron-ion recombination involving the formation of an autoionized state is the product of the probability that the ion-electron system is in the autoionized state and the frequency W_{a0} of transition of the system from this state to the bound states from which the atom later goes over to the ground state. The probability that the atom is in the autoionized state can be found with the technique used for deriving the Saha equation (see Problem 2.7). The system is in a continuous-spectrum state, and the formation of the autoionized state is described by the reaction equation

$$A^+ + e \rightleftarrows A^*$$

Then, using the Boltzmann's formula, we find that the probability that the system is in the autoionized state is

$$\frac{g_a}{g_c} \exp(-\varepsilon_a/T)$$

where g_a is the statistical weight of the autoionized state, g_c is the statistical weight of the continuous-spectrum states given by eq. (2.15), ε_a is the excitation energy of the autoionized state taken from the continuous-spectrum boundary, and T is the electron temperature.

Hence, the recombination frequency is

$$\nu_{rec} = w_{a0} \frac{g_a}{g_c} \exp(-\varepsilon_a/T)$$

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*For instance, the width of the lowest excited autoionized state of He atom, He$(2s^2)^1S$, is about 0.2 eV (the excitation energy of this state is 58 eV). The widths of other autoionized states of helium are considerably smaller. The width of the autoionized state of the helium ion He$^-(1s2s^2)^1S$ is about 0.01 eV. This state gives rise to a resonance in the cross section of elastic scattering of electrons by helium atoms for the electron energy of 19.3 eV.

Using eq. (2.15), we can find now the coefficient of electron-ion recombination via the formation of the autoionized state:

$$\alpha = \frac{\nu_{\text{rec}}}{N_i} = w_{a0}\frac{g_a}{g_e g_i}\left(\frac{2\pi\hbar^2}{mT}\right)^{3/2}\exp\left(-\varepsilon_a/T\right) \tag{4.58}$$

Here $g_e = 2$ and g_i are the statistical weights of the electron and ion, respectively.

When deriving eq. (4.58), we have assumed that the autoionized state is at thermodynamic equilibrium with the continuous spectrum; this is true if the width Γ of the autoionized level is sufficiently large: $\Gamma \gg \hbar w_{a0}$. If this condition is not satisfied, the recombination coefficient can be found from the system of equations (4.55). Owing to frequent transitions between the continuous-spectrum states we shall regard these states as one whose statistical weight, according to eq. (2.15), is $g_c = \frac{g_e g_i}{N_i}\left(\frac{mT}{2\pi\hbar^2}\right)^{3/2}$. Then the system of equations (4.55) is transformed into

$$w_{ca}(\tau_c - \tau_a) = 1$$

$$w_{ac}(\tau_a - \tau_c) + w_{a0}\tau_a = 1$$

Here τ_c and τ_a are the mean times of transition to the bound state from the continuous-spectrum state and the autoionized state, w_{ca} is the frequency of transitions from the continuous-spectrum states to the autoionized state, w_{ac} is the frequency of the inverse transition, and w_{a0} is the frequency of transitions from the autoionized state to the bound state. This system of equations yields the following expression for the recombination time:

$$\tau_c = \left(\frac{1}{w_{ac}} + \frac{1}{w_{a0}}\right)\frac{w_{ac}}{w_{ca}}$$

The principle of detailed balancing yields

$$\frac{w_{ac}}{g_c}\exp\left(-\varepsilon_a/T\right) = \frac{w_{ca}}{g_a}$$

Using it, we obtain the recombination coefficient:

$$\alpha = \frac{1}{\tau_c N_i} = \left(\frac{1}{w_{a0}} + \frac{\hbar}{\Gamma}\right)^{-1}\frac{g_a}{g_e g_i}\left(\frac{2\pi\hbar^2}{mT}\right)^{3/2}\exp\left(-\varepsilon_a/T\right) \tag{4.59}$$

Problem 4.48. Find the coefficient of recombination of electrons and ions via the formation of an autoionized state as a function of the electron temperature.

The cross section of electron-ion recombination proceeding via the formation of an autoionized state is given by the *Breit-Wigner equation*

$$\sigma_{\mathrm{rec}} = \frac{\pi\hbar^2}{2m\varepsilon}\,\frac{\Gamma_{\mathrm{in}}\Gamma}{(\varepsilon-\varepsilon_{\mathrm{a}})^2+\Gamma^2/4}$$

where ε is the energy of the recombining electron, ε_{a} is the excitation energy of the autoionization level, Γ is the width of the autoionization level, $\Gamma_{\mathrm{in}} = \hbar w_{\mathrm{a0}}$ is the inelastic contribution to the level width, corresponding to the transition from the autoionized state to the bound state, and w_{a0} is the frequency of such transitions. Using this equation, we obtain the following expression for the recombination coefficient $\alpha = v\sigma_{\mathrm{rec}}$ averaged over the Maxwellian velocity distribution of electrons (v is the electron velocity):

$$\alpha = \frac{w_{\mathrm{a0}}}{2\pi}\left(\frac{2\pi\hbar^2}{mT}\right)^{3/2}\int_0^\infty \frac{\exp(-\varepsilon/T)\,\Gamma\,d\varepsilon}{(\varepsilon-\varepsilon_{\mathrm{a}})^2+\Gamma^2/4}$$

Here T is the electron temperature.

If the width of the autoionization level tends to zero ($\Gamma \to 0$), the above integral converges for the electron energies $\varepsilon = \varepsilon_{\mathrm{a}}$. This means that recombination is determined by the capture of resonance electrons whose energy is equal to the energy of the autoionization level. The recombination coefficient in this limiting case is

$$\alpha = w_{\mathrm{a0}}\left(\frac{2\pi\hbar^2}{mT}\right)^{3/2}\exp(-\varepsilon_{\mathrm{a}}/T) \tag{4.60a}$$

This equation differs only by the statistical weight from eq. (4.58) which was derived under the assumption of thermodynamic equilibrium between the autoionization level and the continuous spectrum. The difference between the equations is due to the fact that in the Breit-Wigner equation the statistical weights of the ion, electron and atom are taken to be unity.

In the other limiting case of low temperatures the integral in the equation for the recombination coefficient is determined by low electron energies, and we have

$$\alpha = w_{\mathrm{a0}}\left(\frac{2\pi\hbar^2}{mT}\right)^{3/2}\frac{\Gamma T}{2\pi\varepsilon_{\mathrm{a}}^2} \tag{4.60b}$$

Equation (4.60b) is valid for $T/\varepsilon_{\mathrm{a}} \ll 1$, and recombination here is due to the capture of slow electrons.

Comparing eqs. (4.60a) and (4.60b) for different recombination mechanisms we find that eq. (4.60a) is valid if the parameters of the problem satisfy the following condition:

$$\frac{\Gamma}{\varepsilon_{\mathrm{a}}} \ll 2\pi\frac{\varepsilon_{\mathrm{a}}}{T}\exp(-\varepsilon_{\mathrm{a}}/T) \tag{4.61}$$

Equation (4.60b) is valid if the inverse condition is satisfied. Note that according to the definition of the autoionization we have $\Gamma/\varepsilon_a \ll 1$, where $\Gamma = \hbar w_{ac}$ is the width of the autoionization level corresponding to breakdown of this state giving rise to a free electron. If the condition $\Gamma \gg \hbar w_{a0}$ is satisfied, eq. (4.59) coincides with eq. (4.58).

Problem 4.49. Find the coefficient of dissociative recombination of an electron and molecular ion as a function of the electron temperature if the molecular ion is in the ground vibrational state and excited atoms are in one state.

Dissociative recombination of the electron and molecular ion is described by the following reaction equation:

$$e + \mathrm{AB}^+ \to \mathrm{A} + \mathrm{B}^*$$

This process is illustrated by Fig. 4.5 showing the term of the ground state of the molecular ion AB$^+$ and one of the terms of the molecule consisting of the atoms A and B*. These terms intersect when the distance between the nuclei is R_0, and at smaller distances the state of the molecule corresponds to autoionization. The electron is captured to the autoionization level and recombination occurs if the separation distance of the nuclei becomes greater than R_0 before the breakdown of the autoionized state.

According to eq. (4.58), the coefficient of dissociative recombination of the electron and molecular ion in this case is

$$\alpha = \int W(R)\, dR \frac{1}{\tau} \frac{g_a}{2g_i} \left(\frac{2\pi\hbar^2}{mT}\right)^{3/2} \exp\left[-\varepsilon_a(R)/T\right]$$

where $W(R)$ is the probability that the distance between the nuclei is R, so that $\int W(R)\, dR = 1$, τ is the time in which the distance between the nuclei varies from R to R_0, and the width of the autoionization level Γ is much greater than the ratio $\hbar/\tau$.

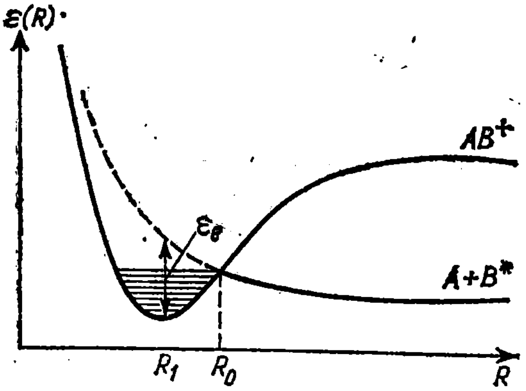

Fig. 4.5. Potential curves for the molecular ion and the molecule in the excited electron state.

Since the molecular ion is in the ground vibrational state, the distance between the nuclei is close to the equilibrium distance R_1, that is, $W(R) = \delta(R - R_1)$. Hence, we obtain

$$\alpha = \frac{1}{\tau(R_1)} \frac{g_a}{2g_i} \left(\frac{2\pi\hbar^2}{mT}\right)^{3/2} \exp\left[-\varepsilon_a(R_1)/T\right]$$

If the condition inverse to condition (4.61) is satisfied, then, according to eq. (4.60 b), we have $\alpha \sim T^{-\frac{1}{2}}$.

Problem 4.50. Find the coefficient of recombination of the positive and negative ions in a high-density gas.

In this case the condition inverse to condition (4.50) is satisfied. The recombining particles spend most of the time approaching each other so that it is convenient to use the following approach to the determination of the recombination coefficient. When the distance between the ions is R the attractive Coulomb force acting on them is e^2/R^2. If the distance R is greater than the mean free path $\lambda \sim 1/N_a\sigma_{el}$ of the ions in the gas, the mean velocity w with which the ions approach each other is equal to the drift velocity in the respective electric field:

$$w = \frac{e}{R^2}(K_+ + K_-)$$

Here K_+ and K_- are the mobilities of the positive and negative ions in the gas. Take a new reference frame in which one of the ions is at rest. Then if the ions of the second species are initially at the distance R from this centre, the number of these ions incident on the centre per unit time is $4\pi R^2 w N_i$, where N_i is the number density of the ions of the second species. Using the equation for the drift velocity, we obtain now the Langevin formula for the coefficient of three-body recombination of the positive and negative ions in a high-density gas:

$$\alpha = 4\pi e(K_+ + K_-) \tag{4.62}$$

Problem 4.51. Find the coefficient of dissociative recombination of an electron and a complex ion as a function of the electron temperature assuming that the electron temperature is not high.

Owing to the complicated nature of the complex ion, the electron energy can be transferred to the internal degrees of freedom of the resulting system via many channels. The respective strong interaction between the electron and the complex ion determines the function we have to find.

We shall employ the following model. Assume that if the electron gets into the region with the radius R_0 around the complex ion, then recombination takes place. Otherwise, no recombination occurs. In this way we have identified the region of strong interaction between the electron and the complex ion. Now we have to find the cross section of electron approach to the complex ion for separation distances smaller than R_0.

Let us use the classical equation of the electron motion to find the relationship between the impact parameter ρ and the closest-approach distance $r_{\min}$. The interaction between the electron and the ion is described by the Coulomb potential; in the range of strong interaction it is considerably higher than the thermal energy. Therefore, the kinetic energy of the electron at the distance $r_{\min}$ from the ion is $e^2/r_{\min}$ and the tangential component of the electron velocity at this distance is $\sqrt{2e^2/mr_{\min}}$ (the normal component of the electron velocity at $r_{\min}$ is zero). Here e is the electron charge and m is the electron mass. Equating the angular momenta of the electron with respect to the ion at a large distance from the ion and at the closest-approach distance, we obtain

$$m\rho v = mr_{\min}\sqrt{\frac{2e^2}{mr_{\min}}}$$

where v is the electron velocity at infinity. Hence we find the relationship between the impact parameter ρ and the closest-approach distance $r_{\min}$. Now we have the recombination cross section which coincides with the cross section of electron getting to a region with the radius R_0 around the ion:

$$\sigma = \pi\rho^2 = 2\pi R_0 \frac{e^2}{mv^2}$$

We have derived this equation using the classical motion equation which is applicable if the main contribution to the cross section is made by collisions with electrons with high orbital angular momentum quantum numbers l for which we have

$$l \sim \frac{m\rho v}{\hbar} \sim \frac{m\sqrt{e^2R_0/mv^2}\,v}{\hbar} \sim \sqrt{\frac{R_0}{a_0}}$$

where $a_0 = \hbar^2/me^2$ is the Bohr radius. Thus, the above result is valid if the size of the region of strong interaction between the electron and the complex ion is considerably greater than characteristic atomic dimensions.

The equation for the recombination cross section yields the recombination coefficient

$$\alpha = \langle v\sigma \rangle = 2\pi\frac{e^2}{m}\left\langle \frac{1}{v} \right\rangle = \frac{2\sqrt{2\pi}e^2}{\sqrt{mT_e}}$$

where the angle brackets denote averaging over the electron velocities, and T_e is the electron temperature. We see that the coefficient of recombination of the electron and the complex ion is inversely proportional to the square root of the electron temperature.

The same result can be derived directly from the Breit-Wigner formula according to which the cross section of recombination of the electron and ion involving the formation of the autoionized state via the channel i is

$$\sigma_i = \frac{\pi\hbar^2(2l+1)}{2m\varepsilon}\cdot\frac{\Gamma_i\Gamma_l}{(\varepsilon-\varepsilon_l)^2+\Gamma_l^2/4}$$

where l is the orbital angular momentum quantum number of the electron, Γ is the total width of the autoionization level, Γ_l is the width of the level due to elastic scattering, Γ_i is the width due to decomposition via the ith channel, ε is the electron energy, and ε_i is the energy of the ith autoionization level.

The total recombination cross section is the sum of the cross sections of recombination via individual channels, and the main contribution to recombination is made by the channels with large widths of autoionization levels. For these channels the second fraction in the expression for the recombination cross section is independent of the electron energy. Thus, the recombination cross sections for the main channels and the total recombination cross section are inversely proportional to the electron energy. Hence, the recombination coefficient is inversely proportional to the square root of the electron temperature as was shown above.

We see that two different methods for taking into account the strong interaction between the electron and the complex ion yield the same result, namely, that the coefficient of dissociative recombination of the electron and the complex ion is inversely proportional to the square root of the electron temperature.

Problem 4.52. Find the recombination coefficient for small variation of the energy of a recombining particle as a result of a single transition.

In this case eq. (4.55) for the recombination time is transformed into a differential equation which can be derived in the following way. Take the Fokker-Planck equation for the probability density as a function of the initial state (see Problem 1.30):

$$-A(\varepsilon)\frac{\partial W(\varepsilon, \varepsilon', t)}{\partial \varepsilon} + B(\varepsilon)\frac{\partial^2 W(\varepsilon, \varepsilon', t)}{\partial \varepsilon^2} = \frac{\partial W(\varepsilon, \varepsilon', t)}{\partial t}$$

The probability $W(\varepsilon, t)$ that a system which is initially in the state with the energy ε reaches the ground state $\left[W(\varepsilon, t) = 1 - \int W(\varepsilon, \varepsilon', t)\rho(\varepsilon')\, d\varepsilon'\right]$ satisfies the equation

$$\frac{\partial(1-v)}{\partial t} = -A\frac{\partial(1-v)}{\partial \varepsilon} + B\frac{\partial^2(1-v)}{\partial \varepsilon^2}$$

Hence, the mean time of recombination is

$$\tau(\varepsilon) = \int_{-\infty}^{\infty} [1 - W(\varepsilon, t)]\, dt = \int_0^{\infty} t\,\frac{\partial W(\varepsilon, t)}{\partial t}\, dt$$

If the system was initially in the state with the energy ε, this time satisfies the equation

$$B\tau'' - A\tau' = -1$$

The factors A and B enter the Fokker-Planck equation (1.71) for the probability density:

$$\frac{\partial(W\rho)}{\partial t}=\frac{\partial}{\partial\varepsilon}\left[A\rho W+\frac{\partial(B\rho W)}{\partial\varepsilon}\right]$$

Since the distribution function has the Maxwellian form at equilibrium [$W=C\exp(-\varepsilon/T)$], the factors A and B are related by

$$A\rho\exp(-\varepsilon/T)=-\frac{d}{d\varepsilon}[B\rho\exp(-\varepsilon/T)]$$

that is,

$$A=-B'+\frac{B}{T}-\frac{\rho'}{\rho}B$$

where T is the temperature of the recombining particles.

Thus, the equation for τ reduces to

$$B\tau''-\left(\frac{B}{T}-B'-B\frac{\rho'}{\rho}\right)\tau'=-1$$

The solution of this equation is

$$\tau(\varepsilon)=\int_{\varepsilon_0}^{\varepsilon}\frac{\exp(x/T)\,dx}{\rho(x)B(x)}\int_{x}^{C_1}\exp(-y/T)\rho(y)\,dy$$

where ε_0 and C_1 are constants.

Now let us consider recombination of electrons and ions. The electron energy will be taken from the boundary of the continuous spectrum, so that $\varepsilon_0=-I$, where I is the ionization potential of the atom. If the energy of the test electron is high (compared with the gas temperature), it only loses energy, that is, the term $B\tau''$ in the equation for τ can be neglected. To satisfy this condition we have to take $C_1=\infty$, and we have

$$\tau(\varepsilon)=\int_{-I}^{\varepsilon}dx\int_{x}^{\infty}\frac{dy\exp(x/T)\,\rho(y)\exp(-y/T)}{B(x)\rho(x)}$$

The region of integration is shown in Fig. 4.6. For the electron-ion recombination the density of states is given by

$$\rho(\varepsilon)=\frac{dn}{d\varepsilon}=\int\delta\left(\varepsilon-\frac{p^2}{2m}+\sum_i\frac{e^2}{r_i}\right)\frac{d\mathbf{p}\,d\mathbf{r}}{(2\pi\hbar)^3}$$

where r_i is the distance from the electron to the ith ion. The density of states can be easily found for $|\varepsilon| \gg e^2 N_i^{1/3}$, where N_i is the number density of ions. If the energy of the electron is positive, its interaction with ions for $\varepsilon \gg e^2 N_i^{1/3}$ can be ignored (because $\sum_i e^2/r_i$ is small compared with ε) and we have

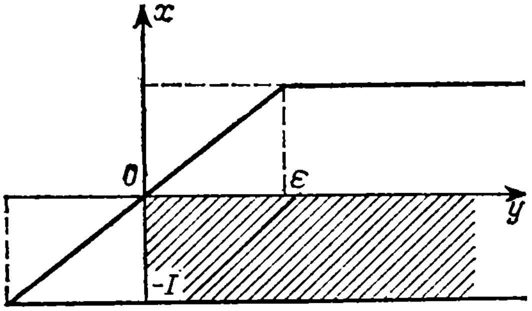

Fig. 4.6. The region of integration in Problem 4.52.

$$\rho_{\rm cont}(\varepsilon) = \int \delta\left(\varepsilon - \frac{p^2}{2m}\right) \frac{d\mathbf{p}\, d\mathbf{r}}{(2\pi\hbar)^3}$$

$$= \frac{4\pi(2m\varepsilon)^{1/2} m\Omega}{(2\pi\hbar)^3}, \qquad \varepsilon \gg e^2 N_i^{1/3}$$

where Ω is the volume of the system, and m is the electron mass. For the bound electron whose binding energy is higher than the potential of interaction of the electron with individual ions ($|\varepsilon| \gg e^2 N_i^{1/3}$) the integral for the density of states can be represented as the sum of the integrals in the vicinity of each electron:

$$\rho_{\rm bound}(\varepsilon) = N_i \Omega \int \delta\left(-|\varepsilon| - \frac{p^2}{2m} + \frac{e^2}{r}\right) \frac{d\mathbf{p}\, d\mathbf{r}}{(2\pi\hbar)^3} = \frac{\sqrt{2}\,\pi^3 e^6 m^{3/2} N_i \Omega}{(2\pi\hbar)^3 |\varepsilon|^{5/2}}$$

$$|\varepsilon \gg e^2 N_i^{1/3}$$

A comparison of the expressions for the densities of states $\rho_{\rm cont}$ and $\rho_{\rm bound}$ shows that for the ideal plasma $\rho_{\rm cont} \gg \rho_{\rm bound}$. Furthermore, for the ideal plasma the energy range $\sim e^2 N_i^{1/3}$, where the density of states is not determined, is small. Since $\rho_{\rm cont}/\rho_{\rm bound} \gg 1$, we can limit the integral in the expression for the recombination time to the shaded region in Fig. 4.6, where $y > 0$ and $x < 0$, and the integrand includes the ratio $\rho_{\rm cont}/\rho_{\rm bound}$. Then we obtain

$$\tau(\varepsilon) \approx \tau(0) = \int_0^I \frac{\exp(-|\varepsilon|/T)\, d|\varepsilon|}{B(\varepsilon)\rho(\varepsilon)} \int_0^\infty \exp(-y/T)\, \rho(y)\, dy$$

$$= \frac{2T^{3/2}}{\pi^{3/2} e^6 N_i} \int_0^\infty \exp(-|\varepsilon|/T)\, |\varepsilon|^{5/2} d|\varepsilon|\, B^{-1}(\varepsilon), \qquad I \gg T \qquad (4.63)$$

Problem 4.53. Calculate the coefficient of electron-ion recombination involving three-body collisions with the gas atoms.

The recombination time is given by eq. (4.63), where we assume that $I \gg T_e$ (T_e is the electron temperature). The problem reduces to finding the quantity

$$B(\varepsilon) = \frac{1}{2}\overline{\frac{\partial \Delta\varepsilon^2}{\partial t}}$$

where the bar above the symbols denotes averaging over electron coordinates and velocities. We shall assume that the motion of the electron in the field of the ion is described by the classical equations and that the electron collides with atoms exchanging energy with them. Then, as shown above (Problem 1.31), we have

$$B(\varepsilon) = \frac{1}{2}\overline{\frac{\partial \Delta\varepsilon^2}{\partial t}} = \frac{1}{3}\, v_a^2 m^2 v_e^2 N_a v_e \sigma^*$$

where v_a is the velocity of the atom with which the electron exchanges energy, v_e is the electron velocity, and σ^* is the diffusion cross section of elastic scattering of the electron by the atom. Average this expression over the atom velocities and the electron velocities. Using the frame of reference in which the ion is at rest, we obtain $\langle v_a^2 \rangle = 3T_g/M + v_i^2$, where T_g is the gas temperature, M is the atomic mass, and v_i is the ion velocity. For low electron velocities we have $\sigma^* = \text{const}$. For the bound electron we have

$$\overline{v_e^3} = \int v_e^3 \delta\left(\varepsilon - \frac{mv_e^2}{2} + \frac{e^2}{r}\right) d\mathbf{v}_e\, d\mathbf{r} \left[\int \delta\left(\varepsilon - \frac{mv_e^2}{2} + \frac{e^2}{r}\right) d\mathbf{v}_e\, d\mathbf{r}\right]^{-1} = \frac{16}{3\pi}\left(\frac{2|\varepsilon|}{m}\right)^{3/2}$$

Hence, we obtain for $\varepsilon < 0$

$$B(\varepsilon) = \frac{T_g}{M}\left(1 + \frac{Mv_i^2}{3T_g}\right) N_a\, \sigma^*\, m^{1/2}\, \frac{16}{3\pi}\, (2|\varepsilon|)^{3/2}$$

We assumed here that the ion velocity does not change after several collisions between the electron and atoms. Hence,

$$\tau = \frac{3MT^{7/2}}{16\sqrt{2\pi}\, e^6 N_i\, T_g \left(1 + \dfrac{Mv_i^2}{3T_g}\right) N_a\, \sigma^* m^{1/2}}$$

The recombination coefficient averaged over the ion velocities is

$$\alpha = \frac{1}{\tau N_i} = \frac{16\sqrt{2\pi}}{3}\, \frac{e^6 N_a}{T_e^{7/2}}\, \frac{T_g\sqrt{m}\,\sigma^*}{\mu} \tag{4.64}$$

where μ is the reduced mass of the atom and the ion.

CHAPTER 5

PROPAGATION OF RADIATION IN GASES

§ 5.1. EQUILIBRIUM OF RADIATION AND GAS

The radiation, which is at thermodynamic equilibrium with the gas occupying the same volume or with the walls of the vessel in which it is contained, is known as the blackbody radiation. The thermodynamic equilibrium of radiation and gas corresponds to strong interaction between them, which is accompanied by absorption and emission of photons. Thermodynamic equilibrium can be found for a narrow range of radiation field frequencies and for entire radiation spectrum. We shall consider below some properties of the blackbody radiation.

Problem 5.1. Derive the expression of the mean number of photons in one state in the case of thermodynamic equilibrium with the temperature T.

The distribution function for the number of photons (oscillators of the radiation field) is determined by the same laws as the distribution functions for other types of oscillations at thermodynamic equilibrium. In particular, by analogy with the distribution of molecules over the vibrational states, we assume that the probability of excitation of n photons in the given state with the frequency ω is $\exp(-\hbar\omega n/T)$. Hence, the mean number of photons in one state with the energy $\hbar\omega$ is

$$\bar{n}_\omega = \frac{\sum\limits_n n \exp\left(-\frac{\hbar\omega n}{T}\right)}{\sum\limits_n \exp\left(-\frac{\hbar\omega n}{T}\right)} = \frac{1}{\exp\left(\frac{\hbar\omega}{T}\right) - 1} \tag{5.1}$$

This equation is known as the Planck distribution.

Problem 5.2. Using the Planck distribution, derive the equation for the spectral density U_ω of blackbody radiation, that is the electromagnetic radiation energy per unit volume and unit frequency range.

Take a volume element V from the entire volume with the radiation field. By definition of the spectral radiation density, the energy of the radiation field in the volume V in the frequency range from ω to $\omega + d\omega$ is

$VU_\omega d\omega$. On the other hand, this energy is $2\dfrac{V d\mathbf{k}}{(2\pi)^3}\hbar\omega n_\omega$, where the factor 2 takes into account the polarization of the transverse electromagnetic wave, $V\,d\mathbf{k}/(2\pi)^3$ is the number of states in the element of the phase space, n_ω is the number of photons in one state, $\mathbf{k}$ is the wave vector of the photon, and $\hbar\omega$ is the energy of the photon. Equating these expressions and using the dispersion relation $\omega = kc$ for the frequency of the electromagnetic wave and the wave vector (where c is the velocity of light), we obtain

$$U_\omega = \frac{\hbar\omega^3}{\pi^2 c^3} n_\omega$$

If the radiation field is at thermodynamic equilibrium with the gas or the walls of the vessel in which it is contained, the number of photons in one state is given by eq. (5.1), and we obtain

$$U_\omega = \frac{\hbar\omega^3}{\pi^2 c^3\left[\exp\left(\frac{\hbar\omega}{T}\right) - 1\right]} \tag{5.2}$$

Equation (5.2) is known as the *Planck radiation formula*. In the limiting case of high frequencies eq. (5.2) reduces to the *Wien law*:

$$U_\omega = \frac{\hbar\omega^3}{\pi^2 c^3}\exp\left(-\frac{\hbar\omega}{T}\right), \quad \frac{\hbar\omega}{T} \gg 1$$

In the limiting case of low frequencies eq. (5.2) reduces to the *Rayleigh-Jeans law*:

$$U_\omega = \frac{\omega^2 T}{\pi^2 c^3}, \quad \frac{\hbar\omega}{T} \ll 1$$

The Rayleigh-Jeans law corresponds to the classical limit, when the result is independent of the quantum parameter $\hbar$. In this case we can use dimensional considerations to find the spectral radiation density. Indeed, in this limiting case the spectral radiation density is expressed in terms of the parameters ω (frequency), T (energy) and c (velocity). These parameters make up only one combination with the dimension of the spectral radiation density, namely, $\omega^2 T c^{-3}$. Thus, we have $U_\omega \sim \omega^2 T c^{-3}$ which differs only by a constant factor from the Rayleigh-Jeans formula.

Problem 5.3. Find the radiation energy flux density for the blackbody radiation (the *Stefan-Boltzmann law*).

The radiation energy density in the frequency range from ω to $\omega + d\omega$ in a volume filled with blackbody radiation is

$$\hbar\omega i_\omega\, d\omega = cU_\omega\, d\omega$$

where i_ω is the flux of photons which is given by the Planck formula:

$$i_\omega = \frac{\omega^2}{\pi^2 c^2}\left[\exp\left(\frac{\hbar\omega}{T}\right) - 1\right]^{-1} \tag{5.3}$$

This photon flux is isotropic in each point of the volume with radiation.

We shall regard the radiation energy flux emitted from the blackbody surface as the radiation flux coming out from a hole in a cavity with opaque walls, which is filled with blackbody radiation. The radiation flux $\int i_\omega \hbar\omega\, d\omega$ is emitted isotropically from the blackbody surface, so that the energy flux emitted in the solid angle element do is $\frac{do}{4\pi}\int i_\omega \hbar\omega\, d\omega$. Projecting the elementary radiation fluxes on the direction of the net flux which is perpendicular to the emitting surface and taking into account only that part of the flux which escapes from the emitting body, we obtain that the net radiation flux is

$$J = \int_0^{\frac{\pi}{2}} \frac{2\pi \cos\theta\, d(\cos\theta)}{4\pi} \int_0^\infty i_\omega \hbar\omega\, d\omega = \frac{1}{4}\int_0^\infty i_\omega \hbar\omega\, d\omega = \sigma T^4 \tag{5.4}$$

where θ is the angle between the normal to the surface and the direction of the emitted photon, and $\sigma = (4\pi^2 c^2 \hbar^3)^{-1} \int_0^\infty (\exp x - 1)^{-1} x^3\, dx = \pi^2/60c^2\hbar^3 = 5.67 \times 10^{-12}$ W·cm^{-2}·K^{-4} is the Stefan-Boltzmann constant. Equation (5.4) is known as the Stefan-Boltzmann law.

Problem 5.4. Derive the Stefan-Boltzmann law using dimensional analysis.

The equilibrium radiation emitted from the blackbody surface is determined by the body temperature T and the radiation parameters $\hbar$ (the Planck's constant) and c (the velocity of light). Only one combination of these parameters with the dimension of the energy flux J can be made up. This combination is

$$J \sim T^4/c^2\hbar^3$$

and differs from the Stefan-Boltzmann law (5.4) only by a constant factor.

Problem 5.5. Derive the equation of the state for the radiation field which is in thermodynamic equilibrium with a gas and is contained in a given volume.

Momentum transported by photons gives rise of the pressure on the gas in the volume or on the walls of the vessel in which it is contained.

Let us calculate the force acting on a unit surface area on both sides in the gas. Each photon with the energy $\hbar\omega$ has the momentum $\hbar\omega/c$, where c is the velocity of light. The flux of photons with the frequencies in the range from ω to $\omega + d\omega$ across the unit surface area is $j_\omega\, d\omega$, so that the pressure of the light at both sides of the unit area is

$$p = \int \hbar\omega j_\omega d\omega$$

For instance, if in the given volume there is the blackbody radiation with the temperature T, the pressure of light is

$$p = \sigma T^4/c$$

where σ is the Stefan-Boltzmann constant.

Problem 5.6. Derive the relationship between the absorption coefficient k_ω in a gas and the probability a_ω of emission of a photon for a given frequency and a given transition.

The coefficient of absorption in the gas is defined by the following relation:

$$\frac{dJ_\omega}{dx} = - k_\omega J_\omega \tag{5.5}$$

Here J_ω is the intensity of the radiation propagating along the x-axis. The relationship between the absorption coefficient k_ω and the frequency distribution a_ω of emitted photons can be found from the condition of thermodynamic equilibrium between the radiation and the gas.

If the radiation and the gas are at thermodynamic equilibrium, the radiation flux in the gas, according to the laws of blackbody radiations, is

$$i_\omega = \frac{\omega^2}{\pi^2 c^2}\left[\exp\left(\frac{\hbar\omega}{T}\right) - 1\right]^{-1}$$

The number of photons absorbed per unit time per unit volume in the frequency range from ω to $\omega + d\omega$ is $k_\omega i_\omega\, d\omega$. At equilibrium this number must be equal to the number of photons emitted per unit time, which is

$$\frac{N_u}{\tau} a_\omega\, d\omega$$

where N_u is the number density of atomic particles on the upper level of the transition, and τ is the lifetime of the atomic particle on the upper level with respect to the transition under consideration. Equating these numbers, we obtain

$$k_\omega = \frac{N_u}{\tau} \frac{a_\omega}{i_\omega}$$

The number densities of the atomic particles on the upper and lower levels of the transition are related by the Boltzmann law:

$$N_u = \frac{N_0}{g_0} g_u \exp(-\hbar\omega/T)$$

Here N_0 is the number density of the atomic particles on the lower transition level, and g_0 and g_u are the statistical weights of the respective states. Using this equation and the expression for the blackbody radiation flux, we obtain

$$k_\omega = \frac{N_0}{\tau} a_\omega \frac{g_u}{g_0} \frac{\pi^2 c^2}{\omega^2} [1 - \exp(-\hbar\omega/T)]$$

This relation is valid at thermodynamic equilibrium between the upper and lower levels of the transition. In the absence of equilibrium the temperature entering this relation must be found from the Boltzmann formula. Hence, we obtain

$$k_\omega = \frac{N_0}{\tau} \frac{g_u}{g_0} a_\omega \frac{\pi^2 c^2}{\omega^2} \left(1 - \frac{N_u g_0}{N_0 g_u}\right) \tag{5.6}$$

This relationship between the parameters describing absorption and emission of photons for a given transition is valid in the absence of thermodynamic equilibrium in a gas.

§ 5.2. PROPAGATION OF THE RESONANCE RADIATION IN A GAS

The lower excited states of atoms or molecules from which effective radiative transition to the ground state can occur are known as the *resonance excited states.* The radiation emitted or absorbed in such transition is known as the *resonance radiation.*

The interaction between a resonance radiation and a gas is highly effective. The resonance radiation can account for the bulk of radiation emitted by a gas. Moreover, the mean free path of the resonance photons (that is, photons of the resonance radiation) is not long. Therefore, the processes of absorption and reradiation of the resonance photons play an important part in the propagation of resonance radiation. Furthermore, resonance radiation can affect the density of excited atoms. All these factors affect the character of propagation of the resonance radiation in a gaseous medium. The reradiation process depends on the shape of the radiation spectral line. It is significant that for low radiation intensities the reradiation of photons is not coherent, that is, the frequency of the emitted photon is independent of the frequency of the photon absorbed with excitation of the atomic particle.

Problem 5.7. For the Lorentz and Doppler line shapes find the probability $P(r)$ that a resonance photon travels a distance r without absorption in an isotropic gaseous medium. Assume that the absorption coefficient is independent of the coordinate and the distance r is much greater than the mean free path of the photons corresponding to the centre of the line.

By definition of the absorption coefficient (5.5), the probability that a photon with the frequency ω travels the distance r without absorption is $\exp(-k_\omega r)$. Since the probability of emission of a photon in the frequency range from ω to $\omega + d\omega$ is $a_\omega d\omega$, the probability that the resonance photon is not absorbed while travelling the distance r is

$$P(r) = \int a_\omega \, d\omega \exp(-k_\omega r) \tag{5.7}$$

Let us calculate this probability for the Lorentz line shape for which $a_\omega = \frac{\nu}{\pi}[(\omega - \omega_0)^2 + \nu^2]^{-1}$, where ν is the width of the line. Using a new variable $s = (\omega - \omega_0)/\nu$, we obtain $a_\omega d\omega = ds/\pi(1 + s^2)$ and $k_\omega = k_0/(1 + s^2)$, where k_0 is the absorption coefficient for the centre of the line. Substitution of these expressions into eq. (5.7) yields

$$P(r) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{ds}{(1 + s^2)} \exp\left(-\frac{k_0 r}{1 + s^2}\right)$$

$$= \int_0^\pi \frac{d\varphi}{\pi} \exp\left(-k_0 r \cos^2 \frac{\varphi}{2}\right) = \exp\left(-\frac{k_0 r}{2}\right) I_0\left(\frac{k_0 r}{2}\right)$$

where $\varphi = 2 \arctan s$, and I_0 is the Bessel function. In the limiting case, when the distance travelled by photons is much greater than the mean free path $1/k_0$ of the photon at the line centre, the equation gives

$$P(r) = (\pi k_0 r)^{-1}, \qquad k_0 r \gg 1$$

For the Doppler line shape make use of eq. (A4.1) for the frequency distribution of emitted photons and introduce a new variable $s = \frac{\omega - \omega_0}{\omega_0} \sqrt{\frac{Mc^2}{2T}}$. We have $a_\omega d\omega = \exp(-s^2)\, ds$ and $k_\omega = k_0 \exp(-s^2)$; then eq. (5.7) yields

$$P(r) = \frac{2}{\sqrt{\pi}} \int_0^\infty \exp(-s^2)\, ds \exp[-k_0 r \exp(-s^2)]$$

Using a new variable $t = k_0 r \exp(-s^2)$, where $0 \leqslant t \leqslant k_0 r$, we transform this expression into

$$P(r) = \frac{1}{\sqrt{\pi k_0 r}} \int_0^{k_0 r} \exp(-t)\, dt \left(\ln \frac{k_0 r}{t} \right)^{-1}$$

Let us evaluate this integral in the limit $k_0 r \gg 1$, when the distance r is much greater than the mean free path of photons at the line centre. Since the main contribution to the integral is made by $t \sim 1$, we can take infinity as the upper limit of the integral. Then we obtain $P(r) = \left(\sqrt{\pi k_0 r} \sqrt{\ln \frac{k_0 r}{t_0}} \right)^{-1}$, where t_0 is given by $\int_0^\infty \exp(-t)\, dt \ln \frac{t}{t_0} = 0$. Finally, we have

$$P(r) = [\sqrt{\pi k_0 r} \sqrt{\ln k_0 r + C}]^{-1}$$

where $C = 0.577$ is the Euler constant.

Problem 5.8. Derive the expression for the flux of resonance photons escaping from a gas volume with the frequency ω and the total flux of photons.

Figure 5.1 illustrates the geometry of the problem. Take as the origin of coordinates the point of the surface at which the escaping radiation is found. Then the radiation flux in the frequency range from ω to $\omega + d\omega$ at this point is

$$j_\omega d\omega = \int \frac{N_u(\mathbf{r})\, d\mathbf{r}}{\tau} a_\omega d\omega \frac{1}{4\pi r^2} \exp\left(- \int_0^r k_\omega dr' \right) \cos\theta \qquad (5.8)$$

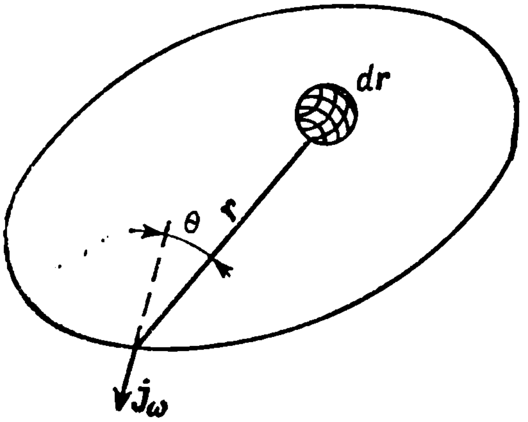

Fig. 5.1. Radiation of a gas volume.

Here the first factor $N_u(\mathbf{r})\, d\mathbf{r}/\tau$ is the number of photons emitted by the volume element $d\mathbf{r}$ per unit time, N_u is the number density of atomic particles in the upper state of the transition, and τ is the lifetime of this state with respect to de-excitation. The second factor describes the probability that photons are emitted in the given frequency range; $4\pi r^2$ is the surface area at the distance r from the source ($\mathbf{r}$ is the distance from the given point of the surface to the volume element $d\mathbf{r}$). Assuming that the distribution of emitted photons is isotropic, we can find the photon flux by dividing the number of photons emitted by the given volume element per unit time by the surface area. The factor $\exp\left(-\int k_\omega\, dr'\right)$ in eq. (5.8) is the probability that the emitted photon has reached the surface without absorption, and $\cos\theta$ accounts for vector summation of the photon fluxes from individual volume elements (θ is the angle between the direction of the total flux and the vector $\mathbf{r}$).

Let us write down the expression for the total photon flux at a given surface point using eq. (5.8). We have

$$j = \int j_\omega\, d\omega = \int \frac{N_u(\mathbf{r})\, d\mathbf{r}}{\tau} \frac{P(r)}{4\pi r^2} \cos\theta \tag{5.9}$$

where $P(r)$ given by eq. (5.7) is the probability that the emitted photon travels the distance r without being absorbed. The probability $P(r)$ is convenient to use since it is independent of the geometry of the problem. Note that eq. (5.9) implies that the vectors $\mathbf{j}_\omega$ have the same direction.

Problem 5.9. The number density N of excited atoms is uniform throughout the gas volume. Calculate the photon flux from the gas if the radiation line has the Lorentz shape, the absorption coefficient for the line centre is k_0 and the lifetime τ of excited atoms with respect to photon emission is considerably longer than the characteristic relaxation time for atom number density. Consider the cases of a gas between two parallel infinite planes at the distance d from each other, in an infinite cylinder with the diameter d, and in a sphere with the diameter d.

At first, let us estimate the photon flux by the order of magnitude. If there were no absorption of photons in the gas volume, the photon flux would be

$$j \sim \frac{N_u}{\tau} d$$

where d is the smallest characteristic dimension of the system. However, the probability that a photon escapes from the system without absorption is $P(d) \sim (k_0 d)^{-1/2}$, as shown in Problem 5.7. Hence, we obtain

$$j \sim \frac{N_u d^{1/2}}{\tau k_0^{1/2}}$$

This dependence of the photon flux on the parameters of the problem is independent of the geometry of the system.

Let us now calculate the photon flux at a given point of the surface. Taking the origin of coordinates at this surface point, we obtain from eq. (5.9)

$$j = \int \frac{N_u(\mathbf{r})\, d\mathbf{r}}{\tau} \frac{P(r)\cos\theta}{4\pi r^2}$$

Owing to the symmetry of this problem, the photon flux is perpendicular to the surface, so that the angle θ is the angle between the normal to the surface and the radius vector between the point of the surface and the point from which the photon is emitted.

In this limiting case the main contribution to the photon flux is made by the region which is at the distance $\sim d \gg 1/k_0$ from the surface. Using the asymptotic expression for the probability $P(r)$, we obtain

$$j = \frac{N_u}{4\pi^{3/2} k_0^{1/2} \tau} \int \frac{d\mathbf{r} \cos\theta}{r^{5/2}}$$

Taking the x-axis perpendicular to the surface, we can write

$$j = j_0 \frac{1}{4\pi^{3/2} d^{1/2}} \int \frac{x\, dx\, dy\, dz}{(x^2 + y^2 + z^2)^{7/4}}$$

where

$$j_0 = \frac{N_u d^{1/2}}{\tau k_0^{1/2}}, \quad r = (x^2 + y^2 + z^2)^{1/2}, \quad \cos\theta = \frac{x}{r}$$

Let us now analyze the specific cases. If the gas is between two infinite walls which are at the distance d from each other, we can define new variables $\rho = \sqrt{y^2 + z^2}$ and $\varphi = \arctan \frac{y}{z}$ and write

$$j = \frac{j_0}{4\pi^{3/2} d^{1/2}} \int_0^d x\, dx \int_0^\infty \frac{2\pi\rho\, d\rho}{(\rho^2 + x^2)^{7/4}} = \frac{j_0}{3\pi^{1/2} d^{1/2}} \int_0^d \frac{dx}{\sqrt{x}} = \frac{2j_0}{3\sqrt{\pi}} \approx 0.376 j_0$$

If the excited gas is in a cylinder with the radius d, we can use new variables $\rho = \sqrt{x^2 + y^2}$ and $\varphi = \arctan \frac{y}{x}$ and obtain after integration over dz

$$j = \frac{j_0}{4\pi^{3/2} d^{1/2}} \int \rho \cos\varphi \cdot \rho\, d\rho\, d\varphi \cdot \frac{\sqrt{\pi}\, \Gamma(5/4)}{\rho^{5/2} \Gamma(7/4)}$$

$$= \frac{j_0}{4\pi d^{1/2}} \frac{\Gamma(5/4)}{\Gamma(7/4)} \int_0^\pi \cos\varphi\, d\varphi \int_0^{d\cos\varphi} \frac{d\rho}{\sqrt{\rho}} = \frac{j_0 \Gamma^2(5/4)}{2\sqrt{\pi}\, \Gamma^2(7/4)} \approx 0.274 j_0$$

If the gas is in a sphere with the diameter d, we can use the spherical coordinates r, θ and φ $(0 \leqslant r \leqslant d \cos\theta,\ 0 \leqslant \theta \leqslant \pi/2,\ 0 \leqslant \varphi \leqslant 2\pi)$ and obtain

$$j = \frac{j_0}{4\pi^{3/2} d^{1/2}} \int_0^1 2\pi d(\cos\theta) \int_0^{d\cos\theta} r^2 dr \frac{r\cos\theta}{r^{7/2}}$$

$$= \frac{j_0}{\sqrt{\pi}} \int_0^1 \cos^{3/2}\theta\, d(\cos\theta) = \frac{2j_0}{5\sqrt{\pi}} \approx 0.226 j_0$$

Now we can write the following approximate expression for the mean photon flux across a surface:

$$j = \frac{0.54 N_u}{\tau k_0^{1/2}} \sqrt{\frac{V}{S}}$$

Here V is the gas volume, and S is the surface area. The factor 0.54 is found from the above three cases the approximate results for which differ from the accurate values by not more than 3%. These approximate results are: $0.382 j_0$ for the slab, $0.270 j_0$ for the cylinder, and $0.220 j_0$ for the sphere.

Let us analyze the case when the photon flux can freely escape from the system. If all emitted photons are not absorbed in the gas system, the mean escaping photon flux is

$$j = \frac{N_u}{\tau} \frac{V}{S}$$

In the above three cases the photon flux is independent of the surface point owing to symmetry.

We see that for the Lorentz line shape in the limit of the photon mean free path at the line centre being much greater than the system dimensions, the emitted photon flux is greater by the factor of $1.85\sqrt{k_0 V/S}$ than the photon flux in the inverse case.

Problem 5.10. Calculate the flux of resonance photons of a given frequency escaping from a gas volume. Assume that the mean free path of the photons is much smaller than the dimensions of the gas system and its radius of curvature. The number densities of the excited and nonexcited atoms are constant at a distance from the surface of the order of the mean free path of the resonance photons.

Let us use eq. (5.8) for the simplified problem conditions. Assume that the boundary surface is flat and the number density N_u of the excited particles and the absorption coefficient k_ω are constant in the volume. Then, owing to the symmetry of the problem, the photon flux is perpendicular to the boundary surface. Taking the origin of the spherical system of coordinates at a given surface point, we obtain from eq. (5.8)

$$j_\omega = \frac{N_u a_\omega}{\tau} \int_0^\infty \int_0^1 \frac{2\pi r^2 \, dr \, d(\cos\theta)}{4\pi r^2} \exp(-k_\omega r) = \frac{N_u a_\omega}{4\tau k_\omega} \tag{5.10}$$

Equation (5.10) gives the photon flux from a gas volume in which equilibrium is maintained between the states of atomic particles and the radiation. Indeed, assume that such equilibrium takes place and the radiation flux with a given frequency in the gas volume is i_ω. Then the number of photons of the given frequency absorbed per unit time per unit volume is $i_\omega k_\omega$ and the number of photons of this frequency emitted per unit time per unit volume is $\frac{N_u}{\tau} a_\omega$. Equating these numbers, we obtain

$$i_\omega = \frac{N_u a_\omega}{\tau k_\omega}$$

The photon flux i_ω in the system has a random spatial distribution, so that the photon flux at a given point with the direction within the solid angle element do is $i_\omega do/4\pi$. To find the flux across the boundary surface, we have to calculate the sum of the normal components of the random photon fluxes at the surface. Hence, the photon flux across the boundary surface is

$$j_\omega = \int_{\cos\theta > 0} \frac{i_\omega \, do}{4\pi} = \frac{i_\omega}{4} = \frac{N_u a_\omega}{4\tau k_\omega}$$

Thus, the equilibrium between absorption and emission of photons of a given frequency near the boundary surface leads to eq. (5.10) for the radiation flux.

Problem 5.11. Calculate the radiation flux for a given frequency, escaping from the infinite flat gas layer with the thickness l. Assume that all the parameters in the layer are determined only by the distance from the surface.

Let us denote the optical thickness by

$$u = \int_0^x k_\omega \, dx$$

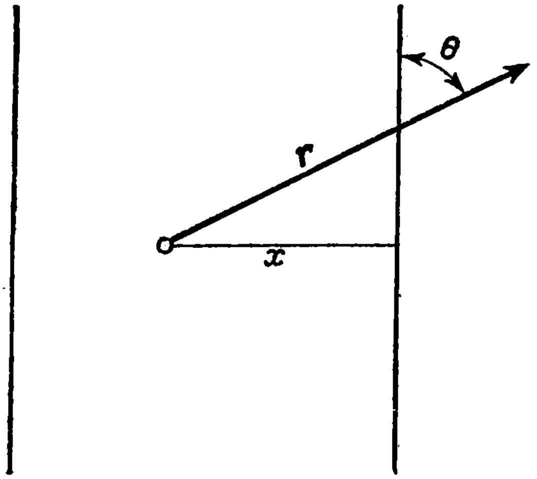

Fig. 5.2. Radiation of a flat gas layer.

where x is the distance to the surface for which the radiation flux is calculated (see Fig. 5.2). Hence, the optical thickness of the entire gas layer is $u_\omega = \int_0^l k_\omega \, dx$. We have $du = k_\omega \, dx = k_\omega \, dr \cos\theta$ and the volume element is $d\mathbf{r} = 2\pi r^2 \, dr \, d(\cos\theta)$. Using the relationship (5.6) between the absorption coefficient k_ω and the frequency distribution a_ω of the emitted photons, we can rewrite eq. (5.8) for the photon flux as

$$j_\omega = \int d(\cos\theta) \, du \frac{N_u a_\omega}{2\pi k_\omega} \exp(-u/\cos\theta)$$

$$= \frac{\omega^2}{2\pi^2 c^2} \int_0^1 d(\cos\theta) \int_0^{u_\omega} du \exp(-u/\cos\theta) \left(\frac{g_u N_0}{g_0 N_u} - 1 \right)^{-1} \qquad (5.11)$$

If the thermodynamic equilibrium is maintained in the gas and the gas temperature is independent of the layer thickness, the Boltzmann distribution gives $g_u N_0 / g_0 N_u = \exp(\hbar\omega/T)$ and eq. (5.11) can be transformed into

$$j_\omega = 2 j_\omega^{(0)} \int_0^1 \cos\theta \, d(\cos\theta) \left[1 - \exp(-u_\omega/\cos\theta) \right]$$

where $j_\omega^{(0)} = \dfrac{\omega^2}{4\pi^2 c^2} [\exp(\hbar\omega/T) - 1]^{-1}$ is the photon flux of a given frequency emitted by the blackbody surface. In the limiting cases we obtain

$$j_\omega = \begin{cases} j_\omega^{(0)} & \text{for } u_\omega \gg 1 \\ 2 j_\omega^{(0)} u_\omega & \text{for } u_\omega \ll 1 \end{cases}$$

Thus, the optical thickness is a convenient parameter for describing radiation outside the system. If the mean free path $1/k_\omega$ of photons is small compared with dimensions of the system, that is, the optical thickness for the given frequency is large, the radiation and the gas in the system are at thermodynamic equilibrium and the photon flux is given by the blackbody equations. In the other limiting case of low optical thickness the photon flux from the surface is proportional to the optical thickness and small compared with the equilibrium photon flux $j_\omega^{(0)}$.

Problem 5.12. Estimate the line width for radiation escaping from a gas volume of the dimension l, assuming the Lorentz and Doppler shapes of the spectral lines emitted by individual gas atoms. Assume that the mean free path $1/k_0$ of photons at the line centre is small compared with the dimensions of the system.

The results of Problem 5.11 show that the radiation flux emitted by the system is that of the blackbody radiation for the frequencies for which the optical thickness of the gas system is large. For the frequencies for which the optical thickness is small, the radiation flux is small compared with the blackbody radiation flux. Hence, the width of lines for radiation emitted by the gas surface is determined by the frequencies for which the optical thickness of the layer is of the order of unity.

For the Lorentz line shape the absorption coefficient at the line wings, according to eq. (A4.2), is $k_\omega = k_0 \nu^2/\pi(\omega - \omega_0)^2$, where ν is the width of the radiation line of an individual atom, and k_0 is the absorption coefficient at the line centre. Hence, the optical thickness of the gas system is $u_\omega \sim k_0 l \nu^2/(\omega - \omega_0)^2$ and the condition $u_\omega \sim 1$ gives the following estimate for the line width:

$$\Delta\omega \sim \nu\sqrt{k_0 l}$$

According to eq. (A4.1), for the Doppler line shape we have $k_\omega = k_0 \exp[-(\omega - \omega_0)^2/\nu_D^2]$, where ν_D is the line width for the radiation of an individual atom in the case of Doppler broadening. The condition $u_\omega \sim k_\omega l \sim 1$ yields the following estimate for the line width:

$$\Delta\omega \sim \nu_D\sqrt{\ln k_0 l}$$

We see that for both line shapes the line width of the escaping radiation is larger than is the width of the line emitted by an individual atom owing to the condition $k_0 l \gg 1$. The dependences of the line width on the optical thickness $k_0 l$ of the system at the line centre are essentially different in these two cases.

Problem 5.13. Calculate the radiation flux from a semi-infinite gas slab whose temperature slowly varies with depth. The optical thickness of the gas slab is assumed to be large.

Equation (5.11) gives for the radiation flux from a slab:

$$j_\omega = \frac{\omega^2}{2\pi^2 c^2} \int_0^1 d(\cos\theta) \int_0^\infty du \exp(-u/\cos\theta) F(u)$$

Here $F(u) = (g_u N_0 / g_0 N_u - 1)^{-1}$. If the gas temperature were independent of the depth, the function $F(u)$ would be independent of u and the radiation flux of the gas would be that of the blackbody with the temperature identical to the gas temperature.

If the temperature slowly varies with the depth, the photons escaping from the gas are emitted from the layers for which the optical thickness is of the order of unity. Therefore, the radiation flux from a gas with a slowly varying temperature is the same as the blackbody radiation flux for the temperature equal to the temperature of the gas layers the optical thickness for which is of the order of unity for the given frequency of radiation.

Let us derive this result in a more rigorous way. Expand the function $F(u)$ in the vicinity of a point u_1:

$$F(u) = F(u_1) + (u - u_1) F'(u_1) + \frac{(u - u_1)^2}{2} F''(u_1)$$

Substitution of this expansion into the expression for the radiation flux yields

$$j_\omega = \frac{\omega^2}{2\pi c^2} \left[\frac{1}{2} F(u_1) + \left(\frac{1}{3} - \frac{u_1}{2} \right) F'(u_1) + \left(\frac{1}{4} - \frac{u_1}{3} + \frac{u_1^2}{4} \right) F''(u_1) \right]$$

Take $u_1 = 2/3$, so that the second term vanishes and we obtain

$$j_\omega = \frac{\omega^2}{4\pi^2 c^2} \left[F\left(u = \frac{2}{3} \right) + \frac{5}{8} F''\left(u = \frac{2}{3} \right) \right]$$

Ignoring the last term, we see that the sought radiation flux is identical to the radiation flux from the blackbody whose temperature is equal to the temperature of the gas slab for which the optical thickness is 2/3.

Let us find the small parameter used for the above expansion. It is

$$\frac{1}{F} \frac{dF}{du} = \frac{\dfrac{dF}{dT} \dfrac{dT}{dx}}{\dfrac{du}{dx} \cdot F} = \frac{d(\ln T)}{k_\omega dx} \frac{\exp(\hbar\omega/T)}{[\exp(\hbar\omega/T) - 1]} \sim \frac{d(\ln T)}{k_\omega dx} \leqslant 1$$

Thus, the above result is valid if the variation of the gas temperature is small at distances comparable with the mean free path of photons in the gas.

Problem 5.14. Find the coefficient of radiative heat transfer due to energy transfer by photons. The gas parameters are independent of the transverse coordinate and the variation of the gas temperature with height is small.

Let us calculate the photon fluxes in both directions at a given height. If we take the origin at the given height and direct the x-axis upwards, the upward photon flux, as shown above, is $j_\omega^+ = \dfrac{\omega^2}{4\pi^2 c^2} F(u = -2/3)$ and the downward photon flux is $j_\omega^- = \dfrac{\omega^2}{4\pi^2 c^2} F(u = 2/3)$. The difference between these fluxes due to variation of the temperature with height is

$$|j_\omega^+ - j_\omega^-| = \frac{\omega^2}{4\pi^2 c^2}\cdot\frac{4}{3}\left|\frac{dF}{du}\right| = \frac{\omega^2}{3\pi^2 c^2}\,\frac{\left|\dfrac{dF}{dT}\right|\left|\dfrac{dT}{dx}\right|}{\dfrac{du}{dx}}$$

$$= \frac{\omega^2}{3\pi^2 c^2}\,\frac{\hbar\omega\left|\dfrac{dT}{dx}\right|}{k_\omega T^2[\exp(\hbar\omega/T) - 1]}$$

Here we used the expression $F = [\exp(\hbar\omega/T) - 1]^{-1}$.

The heat flux due to transport of the photons in the frequency range from ω to $\omega + d\omega$ is $\hbar\omega|j_\omega^+ - j_\omega^-|\,d\omega$. Using the definition of the heat conductivity coefficient $\varkappa\left(q = -\varkappa\dfrac{dT}{dx}\right)$ and the above heat flux $q = \int \hbar\omega|j_\omega^+ - j_\omega^-|\,d\omega$, we obtain the following expression for the coefficient of radiative heat transfer:

$$\varkappa = \frac{1}{3\pi^2 c^2}\int\frac{\omega^2\,d\omega}{k_\omega}\,\frac{\left(\dfrac{\hbar\omega}{T}\right)^2}{[\exp(\hbar\omega/T) - 1]^2}$$

Integration in this expression should be performed only over the frequencies of those photons the mean free paths of which are small compared with the characteristic dimensions of the gas system.

Problem 5.15. Find the conditions for self-reversal of spectral lines.

Self-reversal of a spectral line occurs when the radiation flux from the system is smaller for the line centre frequency than for the neighbouring frequencies and the intensity of radiation of an individual atom decreases with increasing distance from the line centre (see Fig. 5.3).

Self-reversal of spectral lines takes place if the gas temperature inside the system is higher than near its boundary. Since radiation at the line centre is emitted from the gas near the boundary, where the temperature is lower, the radiation flux at the centre of the line is lower than at the wing of the line for which the optical thickness is of the order of unity and the gas temperature is higher. Hence, the curve of the radiation flux as a function of frequency has a dip at the line centre.

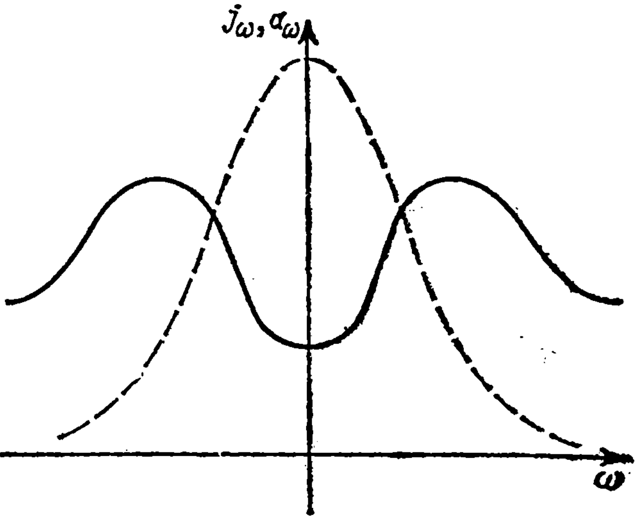

Fig. 5.3. The shape of the spectral line for the elementary radiation event a_ω and for the total radiation flux j_ω escaping from the system in case of self-reversal of the line.

Let us find the conditions under which the radiation flux at the line centre is much lower than at the frequencies corresponding to the maximum radiation flux. According to eq. (5.11), this occurs if the integrand varies noticeably at distances of the order of the mean free path of the photons corresponding to the line centre. Assuming that the distribution of atoms over states is described by the Boltzmann formula and that the number density of excited atoms is much lower than that of nonexcited atoms, we can write down this condition in the following form:

$$\frac{1}{k_0}\frac{\hbar\omega}{T^2}\left|\frac{dT}{dx}\right| \gg 1$$

Problem 5.16. Analyze self-reversal of the radiation line emitted by the infinite gas slab of the thickness l. The gas temperature near the boundary varies as $T(x) = T_0 + \alpha x$, where x is the distance to the slab surface and $\alpha l \ll 1$. The radiation line of the individual atom has the Lorentz shape and the absorption coefficient is spatially uniform. Find the ratio between the maximum photon flux and the photon flux at the line centre.

Use eq. (5.11) for the radiation flux from a gas slab:

$$j_\omega = \frac{\omega^2}{2\pi^2 c^2} \int_0^1 d(\cos\theta) \int_0^{u_\omega} du \exp(-u/\cos\theta)\, F$$

Here $F = (N_0 g_u / N_u g_0 - 1)^{-1} = [\exp(\hbar\omega/T) - 1]^{-1}$. The variation of $F(T)$ in the slab is small and we can approximate it by the expansion $F = F(T_0) + \frac{dF}{dT}\alpha x$, where x is the height of the layer and the second term everywhere is considerably smaller than the first term. The optical thickness of the layer with the height x is related to the optical thickness u_ω of the entire slab for the frequency ω by $u = u_\omega x/l$, since the absorption coefficient is constant in the system. Furthermore, according to eq. (A4.2), for the Lorentz line shape we have $k_\omega = k_0/(1 + s^2)$, where $s = (\omega - \omega_0)/\nu$, ν is the line width, ω_0 is the frequency at the line centre, and k_0 is the absorption coefficient at the line centre. Hence, we obtain

$$j_\omega = 2 j_\omega^{(0)} \int_0^1 d(\cos\theta) \int_0^{u_\omega} du \exp(-u/\cos\theta) \left[1 + \left.\frac{d(\ln F)}{dT}\right|_{T_0} \cdot \frac{\alpha u(1 + s^2)}{k_0} \right]$$

where $j_\omega^{(0)}$ is the blackbody radiation flux for the temperature T_0.

Let us calculate the above integral for the frequencies corresponding to large optical thickness ($u_\omega \gg 1$), finding for the first summand both the main term and the correction and only the main term for the second summand which is much smaller than unity. We obtain

$$j_\omega = j_\omega^{(0)} \left(1 - \frac{\exp(-u_\omega)}{u_\omega} + \frac{a}{u_\omega} \right)$$

where

$$u_\omega = \frac{k_0 l}{1 + s^2}, \quad a = \frac{2}{3} \alpha l \left.\frac{d(\ln F)}{dT}\right|_{T_0} \quad \text{and} \quad a \ll 1$$

Let us analyze this result which is valid for $u_\omega \gg 1$. The optical thickness decreases with increasing distance from the line centre. If $u_0 \gg \ln(1/a)$, where u_0 is the optical thickness for the line centre, then the radiation intensity in the vicinity of the line centre increases with increasing distance from it, that is, self-reversal of the line takes place. The maximum radiation intensity corresponds to the frequency for which the optical thickness satisfies the following condition:

$$(u_\omega + 1) \exp(-u_\omega) = a$$

Since $a \ll 1$, then this condition corresponds to the condition $u_\omega \gg 1$. Thus, the assumptions and equations used here are valid for analyzing the maximum radiation flux which is

$$j_\omega = j_\omega^{(0)}\left(1 + \frac{a}{1 + u_\omega}\right)$$

and since $a \ll 1$ and $u_\omega \gg 1$, the difference between it and the radiation flux at the line centre is small.

Problem 5.17. Derive the kinetic equation for the number density of the atomic particles in the resonance excited state, taking into account the processes of reradiation of photons in a gas. Assume that radiation does not affect transitions between the ground state and the excited state due to collisions.

We have to find the balance equation for the number density of excited atomic particles, taking into account collisions and radiation. If there were no radiation, such balance equation would have the following form:

$$\frac{dN_u}{dt} = -\nu_q N_u + \nu_{exc} N_0$$

Here ν_q is the frequency of transitions from the upper state to the ground state (quenching), and ν_{exc} is the frequency of transitions from the ground state to the upper state due to collisions. Under steady-state conditions $\nu_{exc} = \nu_q N_u^B / N_0^B$, where N_0^B and N_u^B are the number densities at thermodynamic equilibrium, which are given by the Boltzmann formula.

Write down the kinetic equation for the excited particles taking into account radiation in the following form:

$$\frac{\partial N_u(\mathbf{r})}{\partial t} = -\nu_q N_u(\mathbf{r}) + \nu_{exc} N_0(\mathbf{r}) - \frac{N_u(\mathbf{r})}{\tau} + \frac{1}{\tau}\int N_u(\mathbf{r}')\, G(\mathbf{r}', \mathbf{r})\, d\mathbf{r}'$$

The first two terms on the right-hand side of the equation describe collisions of the particles and have the same form as above. The third term describes spontaneous emission of photons by the excited particles, so that τ is the lifetime with respect to spontaneous emission. The last term describes reradiation, that is, absorption of a photon emitted elsewhere. The Green function $G(\mathbf{r}', \mathbf{r})$ has the following physical meaning. The quantity $G(\mathbf{r}', \mathbf{r})\, d\mathbf{r}'$ is the probability that a photon emitted at the point $\mathbf{r}'$ is absorbed at the point $\mathbf{r}$. This function has the following properties: $G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$ and $\int G(\mathbf{r}', \mathbf{r})\, d\mathbf{r}' = 1$. We have used the first property in derivation of the above equation; the second property follows from the definition of the Green function and the integral is taken over infinite space.

Let us find the Green's function for our equation. We have to find the probability that a photon emitted at the point $\mathbf{r}'$ is absorbed in the volume element $d\mathbf{r}$ at the point $\mathbf{r}$ (Fig. 5.4). Write for the volume element $d\mathbf{r} = ds\,dx$, where the x-axis is parallel to the direction of photon propagation and the area element ds is perpendicular to it. Then the probability is

$$G(\mathbf{r}',\mathbf{r})\,d\mathbf{r} = \int a_\omega\,d\omega\,\frac{ds}{4\pi|\mathbf{r}-\mathbf{r}'|^2}\,k_\omega\,dx\,\exp\left(-\int k_\omega\,dx\right)$$

Here the first factor is the probability that the photon frequency is in the given range and the second factor is the probability that the photon travels across the given volume element $d\mathbf{r}$ in case of isotropic photon emission. The third factor is the probability of photon absorption on the path dx, and the fourth factor is the probability that the photon reaches the point $\mathbf{r}$ without absorption. The integral in this factor is taken over the radius between the points $\mathbf{r}$ and $\mathbf{r}'$.

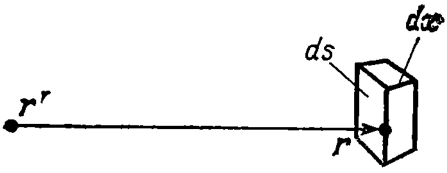

Fig. 5.4. Photon propagation.

The resulting Green function is

$$G(\mathbf{r}',\mathbf{r}) = \int \frac{a_\omega k_\omega\,d\omega}{4\pi|\mathbf{r}-\mathbf{r}'|^2}\exp\left(-\int k_\omega\,dx\right) \tag{5.12}$$

where integration is performed over photon frequencies. Accordingly, the kinetic equation for excited atomic particles is

$$\frac{dN_u(\mathbf{r})}{dt} = -\nu_q N_u(\mathbf{r}) + \nu_{exc}N_0(\mathbf{r}) - \frac{N_u(\mathbf{r})}{\tau} + \frac{1}{\tau}\int \frac{N_u(\mathbf{r}')\,d\mathbf{r}'\,a_\omega k_\omega\,d\omega}{4\pi|\mathbf{r}-\mathbf{r}'|^2}\exp\left(-\int k_\omega\,dx\right) \tag{5.13}$$

This equation is known as the Biberman-Holstein equation.

Problem 5.18. Find the Green function for propagation of radiation in a homogeneous gas for the Lorentz and Doppler line shapes. The distances of propagation are much greater than the mean free path of the photons at the line centre.

First, consider the Lorentz line shape. Using the variable $s = 2(\omega - \omega_0)/\nu$, where ν is the line width [see eq. (A4.2)], we obtain $a_\omega d\omega = ds/\pi(1+s^2)$ and $k_\omega = k_0/(1+s^2)$, where k_0 is the absorption coefficient at the line centre, that is, for the frequency ω_0. Substitution of these expressions into eq. (5.12) for the Green function yields

$$G(R) = \int \frac{a_\omega k_\omega\,d\omega}{4\pi R^2}\exp(-k_\omega R) = \frac{k_0}{4\pi^2 R^2}\int_{-\infty}^{+\infty}\frac{ds}{(1+s^2)^2}\exp\left(-\frac{k_0R}{1+s^2}\right)$$
$$= \frac{k_0}{4\pi R^2}\,\frac{d}{dk_0R}\left[\exp(-k_0R/2)\,I_0\left(\frac{k_0R}{2}\right)\right]$$

where $R = |\mathbf{r} - \mathbf{r}'|$, $I_0(z) = \frac{1}{\pi}\int \exp(- z\cos\varphi)\, d\varphi$, and in calculations we made the replacement $s = \tan\varphi/2$. In the limit $k_0R \gg 1$ this equation yields

$$G(R) = (4\pi)^{-3/2} R^{-7/2} k_0^{-1/2}$$

For the Doppler line shape described by eq. (A4.1) we shall use the dimensionless variable $s = \frac{\omega - \omega_0}{\omega_0}\sqrt{\frac{Mc^2}{2T}}$, so that $a_\omega\, d\omega = \frac{1}{\sqrt{\pi}} \exp(- s^2)\, ds$ and $k_\omega = k_0 \exp(- s^2)$. Substitution of these expressions into eq. (5.12) gives

$$G(R) = \frac{k_0}{4\pi^{3/2} R^2} \int\limits_{-\infty}^{+\infty} \exp(- 2s^2)\, ds \exp(- t) = \int\limits_0^{k_0R} \frac{t\exp(- t)\, dt}{4\pi^{3/2} k_0 R^4 \sqrt{\ln(k_0R/t)}}$$

where $t = k_0R\exp(- s^2)$. In the limit $k_0R \gg 1$, when $\ln(k_0R) \gg 1$, we have

$$G(R) = [4\pi^{3/2} k_0 R^4 \sqrt{\ln(k_0R/t_0)}]^{-1}$$

where t_0 is the solution of the equation $\int\limits_0^\infty t\exp(- t)\, dt \ln(t/t_0) = 0$. Solving this equation, we obtain $\ln t_0 = 1 - C = 0.423$, where C is the Euler constant. Finally, we have

$$G(R) = (4\pi^{3/2} k_0 R^4 \sqrt{\ln k_0R - 0.42})^{-1}$$

Problem 5.19. Find the conditions under which radiation does not disturb the thermodynamic equilibrium between the ground and excited states of atomic particles. Assume that the number density of the excited particles is much lower than the number density of particles in the ground state.

The sought conditions can be found from eq. (5.13) which takes into account the transitions between the states due to collisions and under the effect of radiation. In this problem these processes do not affect the number density of particles in the ground state since practically all the particles are in the ground state. Analysis of the balance equation (5.13) will be facilitated by writing it in terms of dimensionless variables.

Let us define the dimensionless function $y(\mathbf{r}) = N_\mathrm{u}/N_\mathrm{u}^\mathrm{B}$ which equals unity if the number density of excited particles is the same as in the absence of radiation (N_u^B is the number density of excited particles given by the Boltzmann distribution). Other dimensionless variables are $\beta = \nu_\mathrm{q}\tau$ and the time $t' = t/\tau$. Since $N_0\nu_\mathrm{exc} = N_\mathrm{u}^\mathrm{B}\nu_\mathrm{q}$, eq. (5.13) is transformed into

$$\frac{dy(\mathbf{r})}{dt'} = \int y(\mathbf{r}')\, G(\mathbf{r}, \mathbf{r}')\, d\mathbf{r}' + \beta - (1 + \beta)\, y(\mathbf{r}) \qquad (5.14)$$

Equation (5.14) shows that the effect of radiation on the gas properties is described by the parameter β. For instance, when $\beta \gg 1$, we have $y(\mathbf{r}) = 1$, that is, the distribution is identical to that in the absence of radiation. In this limiting case collision processes are fast and radiation does not disturb equilibrium established by collisions of particles.

Thermodynamic equilibrium can be maintained even if the above condition is not satisfied. Let us denote by $W = \int G(\mathbf{r}, \mathbf{r}')\, d\mathbf{r}'$ the probability that a photon emitted in a gas system is absorbed within the system. The value of the dimensionless function y averaged over the system's volume Ω is $\bar{y} = \frac{1}{\Omega} \int y(\mathbf{r})\, d\mathbf{r}$. The approximate solution of the steady-state equation (5.14) after integration over the volume is $\bar{y} = \beta/(1 + \beta - W)$. We see that if the mean free path of photon is small compared with the dimensions of the system ($W \approx 1$), thermodynamic equilibrium is maintained in the main part of the gas system. Indeed, in this case practically all photons are absorbed within the system, that is, radiation does not transfer excitation energy outside the system.

Problem 5.20. Find the number density of excited particles and the radiation flux emitted by the system assuming that the number density of excited particles and other parameters of the gas are constant and uniform. The radiation line has the Lorentz shape and the mean free path of photons at the line centre is small compared with the dimensions of the system.

Since $y(\mathbf{r}) = \text{const}$, in this case we obtain the following solution of the steady-state equation (5.14):

$$y = \frac{\beta}{\beta + 1 - \int G(\mathbf{r}, \mathbf{r}')\, d\mathbf{r}'}$$

The integral here is taken over the gas volume.

The quantity $\int G(\mathbf{r}, \mathbf{r}')\, d\mathbf{r}'$ is the probability that a photon emitted in the gas is absorbed within the gas volume. Let us first assume that the optical thickness of the system is small, that is, an emitted photon can readily escape from the system, so that $\int G(\mathbf{r}, \mathbf{r}')\, d\mathbf{r}' \ll 1$. Then we have $y = \beta/(1 + \beta)$. Hence, we see from the definition of the parameter β that if the characteristic time of quenching of the excited state $1/\nu_q$ is much shorter than the characteristic radiation time τ, we have $y = 1$. In this case radiation does not disturb the equilibrium distribution of the particles over the states.

If the optical thickness of the system is high, that is, if the mean free path of photons at the line centre is much smaller than the dimensions of the system, we shall use a new dimensionless parameter. Denote by $\tau_{\text{eff}} = \tau \Big/ \left[1 - \int G(\mathbf{r}, \mathbf{r}')\, d\mathbf{r}'\right]$ the mean time in which the excitation energy is transferred from the system in the form of radiation. Writing

$$\beta_{\text{eff}} = \nu_q \tau_{\text{eff}} = \frac{\beta}{1 - \int G(\mathbf{r}, \mathbf{r}')\, d\mathbf{r}'}$$

we can write the solution of the steady-state equation (5.14) in the form $y = \beta_{\text{eff}}/(1 + \beta_{\text{eff}})$. We see that in systems with reradiation β_{eff} is a parameter describing the effect of radiation on the optical properties of the system.

Let us calculate the intensity of radiation escaping from the system. The number of photons escaping from the system per unit time is $N_u V/\tau_{\text{eff}}$, where V is the volume of the system. In the limit $\beta_{\text{eff}} \ll 1$ we have $y = \beta_{\text{eff}}$, $N_u = N_u^B \beta_{\text{eff}}$ and the number of photons emitted per unit time is

$$N_u^B \nu_q \nu_{\text{eff}} V/\tau_{\text{eff}} = N_0 \nu_{\text{exc}} V$$

Here we used the above relationship between the frequencies of excitation and quenching. We see that in this limiting case disappearance of the excited states is determined by radiation. Therefore, the total number of photons escaping from the system equals the total number of excited states generated by collision processes.

In the other limiting case, when $\beta_{\text{eff}} \gg 1$, we have $N_u = N_u^B$. The results of Problem 5.9 show that for the Lorentz line shape the total number of photons escaping from the system is

$$jS = \frac{0.54 N_u^B}{\tau\sqrt{k_0}} \sqrt{VS}, \qquad k_0 \frac{V}{S} \gg 1$$

where j is the photon flux, V is the volume of the system, and S is its boundary surface area.

Problem 5.21. The flux j_0 of resonance photons is perpendicular to an infinite slab of plasma with a large optical thickness. Find the number density of excited atoms at the plasma boundary assuming that the electron number density is low, so that each excited state disappears owing to the emission of a photon.

There are two photon fluxes in the plasma. The directed photon flux $j_+(x)$ is produced by the external source (x is the distance to the plasma boundary). The isotropic photon flux $j^*(x)$ is due to reradiation of photons. If the optical thickness of plasma is large, the effect of the second plasma boundary can be neglected. Then the equilibrium condition yields the balance equation for the photon fluxes at the plasma surface:

$$j_0 \equiv j_+ = \frac{1}{2} j^*(0)$$

The number density of excited atoms at the plasma boundary N_u can be found from the balance equation for the number of photons emitted from the unit volume per unit time, which is N_u/τ, and the number of photons absorbed in the unit volume per unit time, which is $k(j_+ + j^*)$, where k is the absorption coefficient. We obtain

$$N_u = 3kj_0\tau$$

§ 5.3. PROPAGATION OF INFRARED RADIATION IN A MOLECULAR GAS

The peculiar features of propagation of infrared radiation in a molecular gas are determined by the fact that the emission spectrum of such gases consists of a large number of lines corresponding to respective vibrational-rotational or rotational transitions. If the line width is considerably smaller than the interval between neighbouring lines, then the absorption coefficient of a molecular gas is a rapidly varying function in a small frequency range. We shall analyze below the propagation of infrared radiation in slabs of a molecular gas or escape of radiation from the gas. We shall use the following functions and parameters.

The absorption function is defined as

$$A(\omega_0, \Delta\omega) = \frac{1}{\Delta\omega} \int\limits_{\omega_0 - \frac{\Delta\omega}{2}}^{\omega_0 + \frac{\Delta\omega}{2}} [1 - \exp(-u_\omega)]\, d\omega \qquad (5.15)$$

where u_ω is the optical thickness of the slab of the molecular gas, and $\Delta\omega$ is the width of the frequency range.

According to eq. (5.15), the absorption function is the mean probability that the radiation in the given frequency range is absorbed in the gas slab. If the given frequency range $\Delta\omega$ is large compared with the frequency difference between neighbouring transitions but the variation of the mean line intensity in this frequency range is small, then the dependence of the absorption function on the magnitude of the given frequency range is weak. In this case the absorption function is a convenient characteristic for describing the propagation of radiation in the gas slab. The quantity

$$1 - A = \frac{1}{\Delta\omega} \int\limits_{\omega_0 - \frac{\Delta\omega}{2}}^{\omega_0 + \frac{\Delta\omega}{2}} \exp(-u_\omega)\, d\omega \qquad (5.16)$$

is known as the transmission function.

The total radiation flux from the surface of the molecular gas slab is

$$j = \int j_\omega\, d\omega$$

where j_ω is the radiation flux for the given frequency. If the gas temperature is constant and the frequency range of radiation is narrow compared with the emitted frequency, then according to eq. (5.11) the total radiation flux due to the given transitions can be written as

$$j = j_\omega^{(0)} \Delta\omega$$

$$\Delta\omega = 2 \int\limits_{-\infty}^{+\infty} d\omega \int\limits_0^1 d(\cos\theta)\,[1 - \exp(-u_\omega/\cos\theta)]\cos\theta \qquad (5.17)$$

where $j_\omega^{(0)}$ is the blackbody radiation flux at the mean transition frequency for the given temperature, and $\Delta\omega$ is the effective frequency range corresponding to the given series of transitions in the gas. If there are several series of radiative transitions and their respective frequency ranges do not overlap, the total flux of the emitted infrared radiation is the sum of the fluxes corresponding to these series of transitions (bands).

The absorption coefficient of the molecular gas in the infrared spectral region is determined by the vibrational-rotational or rotational transitions and is given by

$$k_\omega = \sum_i S_i a(\omega - \omega_i) \qquad (5.18)$$

where ω_i is the frequency of the centre of the line of the respective vibrational-rotational transition, S_i is the intensity of the transition [according to eq. (5.6),

$$S_i = \frac{\pi^2 c^2}{\omega_i^2 \tau_i} N_0[1 - \exp(-\hbar\omega_i/T)]$$

where ω_i is the frequency of the ith transition, ω is the frequency of the radiative transition, and N_0 is the number density of molecules in the lower level of the transition], and $a(\omega - \omega_i)$ is the distribution of the emitted photons over frequencies $\left(\int a(\omega - \omega_i)\, d\omega = 1\right)$.

The expressions (5.18), (5.15) and (5.17) for the absorption coefficient, the absorption function and the radiation flux are too complicated for accurate determination. Therefore, analysis of emission and absorption by molecular gas slabs employs various models and approximations.

1. The model of single lines. This model corresponds to the case when the minimum optical density in the interval between two lines is $u_{\min} \ll 1$, while the optical density at the centre of the respective line is $u_{\max} \gtrsim 1$. Then the frequency range in which the gas slab emits or absorbs radiation consists of the tops of the individual lines for which $u_\omega \gtrsim 1$. This frequency range $\Delta\omega$ for an individual line can be found from the relationship $u_{\max} \dfrac{a(\Delta\omega)}{a(0)} \sim 1$. Accordingly, the absorption function in the given frequency is $A \sim \Delta\omega/\delta$, where δ is the mean interval between the levels.

2. The regular model (the Elsasser model). In this model the lines are assumed to be regularly spaced and their intensities to vary slowly with the variation of the transition number, so that eq. (5.18) can be rewritten as

$$k_\omega = S(\omega) \sum_n a(\omega - \omega_0 \pm n\delta) \tag{5.19}$$

where ω_0 is the frequency of the line centre, and δ is the frequency difference between neighbouring transitions.

This model is valid for linear molecules.

3. The random model. This model assumes random spacing of the lines but the mean interval δ between the neighbouring lines in the given frequency range is predetermined. Furthermore, the probabilistic distribution of line intensities is specified for the given frequency range.

Problem 5.22. Derive the mean absorption function in the given frequency range proceeding from the random model.

Denote the mean interval between the levels by δ and the probability that the intensity of an indivdual line in the given frequency range is in the range from S to $S + dS$ by $p(S)\, dS$. The absorption function for the frequency range containing N lines, as given by eqs. (5.15) and (5.18), is

$$\bar{A} = 1 - \frac{1}{N\delta} \int\limits_{\omega_0 - \frac{N\delta}{2}}^{\omega_0 + \frac{N\delta}{2}} \exp\left[-\sum_{i=1}^{N} u_\omega^{(i)}\right] d\omega$$

where ω_0 is the centre of the given frequency range, and $u_\omega^{(i)}$ is the optical thickness of the gas at the given frequency due to the ith transition.

The random model shows that the probability that the centre of the ith line is in the frequency range from ω_i to $\omega_i + d\omega_i$ is proportional to the width $d\omega_i$ of this range. Then the above formula can be transformed into

$$\bar{A}=1-\frac{\int \exp\left[-\sum_{i=1}^{N} u_{\omega}^{(i)}\right]\prod_{i} d\omega_i}{\prod_{i}\int d\omega_i}=1-\frac{\prod_{i=1}^{N}\int_{\omega_0-\frac{N\delta}{2}}^{\omega_0+\frac{N\delta}{2}} d\omega_i \exp(-u_{\omega}^{(i)})}{\prod_{i=1}^{N}\int_{\omega_0-\frac{N\delta}{2}}^{\omega_0+\frac{N\delta}{2}} d\omega_i}$$

where $\bar{A}$ is the mean probability that a photon in the given frequency range is absorbed by the molecular gas. Since all the frequency ranges on the right-hand side of the equation are identical, we can rewrite it as

$$\bar{A}=1-\left(\frac{1}{N\delta}\int_{\omega_0-\frac{N\delta}{2}}^{\omega_0+\frac{N\delta}{2}} \exp(-u_{\omega})\, d\omega\right)^{N}$$

where u_{ω} is the optical thickness for an individual line.

The width of the given frequency range is large compared with the line width. Then the optical thickness of the gas at the ends of the given frequency range tends to zero and the integral in the above equation tends to unity. Therefore, this equation can be transformed into

$$\bar{A}=1-\left[1-\frac{1}{N\delta}\int_{\omega_0-\frac{N\delta}{2}}^{\omega_0+\frac{N\delta}{2}} [1-\exp(-u_{\omega})]\, d\omega\right]^{N}$$

Tending the number of lines to infinity and noting that the integral is independent of the width of the given frequency range, we obtain

$$\bar{A}=1-\exp\left\{-\int[1-\exp(-u_{\omega})]\frac{d\omega}{\delta}\right\} \tag{5.20}$$

Using the intensity distribution for the lines in the given frequency range, we obtain finally

$$\bar{A}=1-\exp\left\{-\int[1-\exp(-u_{\omega}(S_i))]\frac{d\omega_i}{\delta}p(S_i)\, dS_i\right\} \tag{5.21}$$

where, as above, u_ω is the optical thickness of the gas corresponding to the frequency of the ith transition and the line intensity S_i.

Problem 5.23. Find the absorption coefficient for a frequency band if the mean frequency difference between neighbouring transitions is small compared with the width for an individual transition.

In this case the dependence of the absorption coefficient on the relative positions of neighbouring lines is weak and we shall make use of the random model. Let us average eq. (5.18) for the absorption coefficient over the positions of individual lines. Take an imaginary frequency range $N\delta$ containing N lines (here δ is the mean interval between the lines). The probability that the centre of the ith line is in the frequency range from ω_i to $\omega_i + d\omega_i$ is proportional to $d\omega_i$, so that the mean absorption coefficient in the given frequency range is

$$k_\omega = \sum_{i=1}^{N} \int S_i a(\omega - \omega_i) \frac{d\omega_i}{N\delta} = \frac{1}{N\delta} \sum_{i=1}^{N} S_i = \frac{\bar{S}}{\delta}$$

where $\bar{S}$ is the mean intensity of the lines in the given frequency range.

Problem 5.24. Find the distribution over optical thicknesses in the given frequency range proceeding from the random model.

Denote the mean frequency difference between neighbouring lines by δ and the probability that the line intensity in the given frequency range is between S and $S + dS$ by $p(S)\,dS$. We have to find the probability $f(u)\,du$ that the optical thickness for the given frequency is in the range from u to $u + du$. The specified optical thickness for the given frequency is determined by the random positions of individual lines. The distribution of lines in the given frequency range determines the distribution over optical thicknesses for the given frequency.

We shall employ the method developed for determining the line shape in the statistical theory of line broadening. Define the characteristic function

$$\chi(t) = \int_0^\infty \exp(-itu) f(u)\, du = \prod_i \int \exp(itu_i)\, W(u_i)\, du_i = \prod_i \chi_i(t)$$

where u_i is the optical thickness corresponding to the ith line for the given frequency ($u = \sum_i u_i$), $W(u_i)\, du_i$ is the probability that the optical thickness is in the range from u_i to $u_i + du_i$, so that $f(u)\, du = \int \prod_i W(u_i)\, du_i$. Hence,

$$f(u) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \chi(t) \exp(itu)\, dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \prod_i \chi_i(t) \exp(itu_i)\, dt$$

Consider the imaginary frequency range $\Delta\omega$ containing the given frequency. The probability that the centre of the ith line is between ω_i and $\omega_i + d\omega_i$ in the given frequency range is $d\omega_i/\Delta\omega$. Assuming that the line width is small compared with $\Delta\omega$, so that the optical thickness corresponding to the ith line at the distance of the order of $\Delta\omega$ from its centre is much smaller than unity, we obtain

$$\chi_i(t) = \int \exp(-itu_i)\frac{d\omega_i}{\Delta\omega} = 1 - \frac{1}{\Delta\omega}\int \{1 - \exp[-itu_i(\omega_i)]\}\, d\omega_i$$

Since the second term is small compared with unity, we obtain

$$\ln \chi_i(t) = \frac{1}{\Delta\omega}\int d\omega_i \{1 - \exp[-itu_i(\omega_i)]\}$$

and

$$\ln \chi(t) = \frac{1}{\Delta\omega}\sum_i \int d\omega_i\{1 - \exp[-itu_i(\omega_i)]\}$$

Tending $\Delta\omega$ to infinity and taking into account random distribution of the line positions and identical distributions of intensities, we obtain

$$\ln \chi(t) = \int \frac{d\omega}{\delta}\{1 - \exp[-itu_i(\omega - \omega_i, S)]\}p(S)\, dS$$

The given frequency range $\Delta\omega$ contains N lines, so that $\delta = \Delta\omega/N$ is the mean frequency difference between the neighbouring lines; u_i is the optical thickness corresponding to the line with the centre at the point ω_i and the intensity S. The above equations yield the following distribution function:

$$f(u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt \exp\left\{itu - \int \frac{d\omega_i}{\delta}\, p(S)\, dS[1 - \exp(-itu_i(\omega - \omega_i, S))]\right\} \tag{5.22}$$

Let us find the mean absorption function for the given frequency from eq. (5.22). By definition, we have

$$\bar{A} = \int_0^\infty [1 - \exp(-u)]f(u)\, du = 1 - \int_0^\infty \exp(-u)\, f(u)\, du$$

Calculations yield

$$\bar{A} = 1 - \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{dt}{(1 - it)} \exp\left\{-\int \frac{d\omega_i}{\delta}\, p(S)\, dS\, [1 - \exp(-itu_i)]\right\}$$

The integrand has a pole at the point $t = -i$. Taking the residue of the function at the pole, we obtain finally

$$\bar{A} = 1 - \exp\left\{-\int \frac{d\omega_i}{\delta} p(S)\, dS [1 - \exp(-u_i)]\right\}$$

This expression coincides with the result of Problem 5.22.

Problem 5.25. Find the width of the emission band for a slab of a molecular gas, proceeding from the random model and assuming that the line intensity is determined by its frequency range.

The definition of the width of the emission band yields

$$\Delta\omega = 2\int d\omega \int \cos\theta\, d(\cos\theta)\left[1 - \exp\left(-\frac{u(\omega)}{\cos\theta}\right)\right]$$

Average the integrand over the optical thicknesses for the given frequency:

$$\overline{1 - \exp\left(-\frac{u}{\cos\theta}\right)} = \int_0^\infty f(u)\,du\left[1 - \exp\left(-\frac{u}{\cos\theta}\right)\right]$$

Using the distribution over the optical thicknesses derived in Problem 5.24, we obtain

$$\overline{1 - \exp\left(-\frac{u}{\cos\theta}\right)}$$
$$= 1 - \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt \int_0^\infty du \exp\left\{-\frac{u}{\cos\theta} + itu - \int \frac{d\omega_i}{\delta}[1 - \exp(-itu_i)]\right\}$$

where u_i is the optical thickness corresponding to an individual line.

Integrate this equation over du and find the integral over dt from the residue of the integrand at the pole. We obtain

$$\overline{1 - \exp\left(-\frac{u}{\cos\theta}\right)} = 1 - \exp\left\{-\int \frac{d\omega_i}{\delta}\left[1 - \exp\left(-\frac{u_i}{\cos\theta}\right)\right]\right\}$$

Hence, the width of the emission band found from the random model is

$$\Delta\omega = 2\int_{-\infty}^{+\infty} d\omega \int_0^1 \cos\theta\, d(\cos\theta)\left\{1 - \exp\left[-\int \frac{d\omega_i}{\delta}\left\{1 - \exp\left(-\frac{u_i}{\cos\theta}\right)\right\}\right]\right\}$$

where δ is the mean frequency difference between the neighbouring lines. The integral over $d\omega_i$ determined by the line shape depends on the line intensity as a parameter, while the intensity itself depends on the line frequency ω.

If the line intensity distribution for the given frequency is not random, the above equation should be rewritten as

$$\Delta\omega = 2\int_{-\infty}^{+\infty} d\omega \int_{0}^{1} \cos\theta\, d(\cos\theta)$$
$$\times \left\{1 - \exp\left[-\int p(S)\,dS\,\frac{d\omega_i}{\delta}\left\{1 - \exp\left(-\frac{u_i}{\cos\theta}\right)\right\}\right]\right\}$$

Problem 5.26. Using the random model, calculate the emission band width for a slab of molecular gas assuming that the radiation lines have the Lorentz shape. Assume that the line intensity sharply decreases with increasing distance from the centre of the emission frequency band.

For the Lorentz line shape the optical thickness corresponding to a single line frequency is

$$u = u_{\max}\frac{\nu^2}{(\omega - \omega_i)^2 + \nu^2} = \frac{u_{\max}}{1 + s^2}$$

where ω_i is the frequency of the line centre, $u_{\max}$ is the optical thickness of the gas slab corresponding to the line centre, ν is the line width, and $s = (\omega - \omega_i)/\nu$. Substituting this expression into the equation for the emission band width derived in Problem 5.25 from the random model, we obtain

$$\Delta\omega = 2\int_{-\infty}^{+\infty} d\omega \int_{0}^{1} \cos\theta\, d(\cos\theta)$$
$$\times \left\{1 - \exp\left(-\frac{\nu}{\delta}\int_{-\infty}^{+\infty} ds\left[1 - \exp\left\{-\frac{u_{\max}(\omega)}{(1 + s^2)\cos\theta}\right\}\right]\right)\right\}$$

where $u_{\max}(\omega)$ is the optical thickness of the gas slab for the line centre frequency which is proportional to the line intensity which, in its turn, depends on the transition frequency.

Let us now calculate the emission band width, assuming that $u_{\max}$ sharply decreases with increasing distance from the centre of the band. Note that the main contribution to the integral is made by the frequencies for which

$$\frac{\nu}{\delta}\int ds\left\{1 - \exp\left[-\frac{u_{\max}}{(1 + s^2)\cos\theta}\right]\right\} \sim 1$$

At first, let us assume that the line width ν is small compared with the frequency difference between the neighbouring lines. Then the main contribution to the integral is made by the frequencies whose distance from the line centre is considerably greater than the line width, so that

$$\frac{\nu}{\delta}\int\limits_{-\infty}^{+\infty} ds\left[1-\exp\left(-\frac{u_{\max}}{s^2\cos\theta}\right)\right]=2\frac{\nu}{\delta}\sqrt{\frac{\pi u_{\max}}{\cos\theta}}$$

and the emission band width is

$$\Delta\omega=2\int\limits_{-\infty}^{+\infty} d\omega\int\limits_{0}^{1}\cos\theta\, d(\cos\theta)\left\{1-\exp\left[-\frac{2\nu}{\delta}\sqrt{\frac{\pi u_{\max}(\omega)}{\cos\theta}}\right]\right\}$$

Here we should take into account the fact that $u_{\max}$ is strongly dependent on frequency. The main contribution to the integral is made by the frequencies for which the exponent is of the order of unity. But since $u_{\max}$ rapidly varies with frequency, the term in the braces practically varies from zero to unity in a narrow frequency range. Therefore, let us take a frequency ω' close to the band centre frequency ω_0 but for which the exponent is much greater than unity. In the vicinity of ω' the frequency dependence of the optical thickness at the line centre can be written as $u_{\max}(\omega)=u_{\max}(\omega')\exp[-\alpha(\omega-\omega')]$, where α is the logarithmic derivative of $u_{\max}$ for the frequency ω'. Hence,

$$\Delta\omega=2(\omega'-\omega_0)$$
$$+4\int\limits_{\omega'}^{\infty} d\omega\int\limits_{0}^{1}\cos\theta\, d(\cos\theta)\left\{1-\exp\left[-\frac{2\nu}{\delta}\sqrt{\frac{\pi u_{\max}(\omega')}{\cos\theta}}\exp\left(-\frac{\alpha(\omega-\omega')}{2}\right)\right]\right\}$$

Replacement of variables yields

$$\Delta\omega=2(\omega'-\omega_0)+\frac{8}{\alpha}\int\limits_{0}^{1}\cos\theta\, d(\cos\theta)\int\limits_{0}^{A/\sqrt{\cos\theta}}[1-\exp(-z)]\frac{dz}{z}$$

where

$$z=\frac{A}{\sqrt{\cos\theta}}\exp\left(-\frac{\alpha(\omega-\omega')}{2}\right)\quad\text{and}\quad A=\frac{2\nu}{\delta}\sqrt{\pi u_{\max}(\omega')}\gg 1$$

Since the upper limit for the last integral is considerably greater than unity, we can use the asymptotic expression for it and obtain

$$\Delta\omega=2(\omega'-\omega_0)+\frac{8}{\alpha}\int\limits_{0}^{1}\cos\theta\, d(\cos\theta)\ln\frac{A\exp(C)}{\sqrt{\cos\theta}}$$
$$=2(\omega'-\omega_0)+\frac{4}{\alpha}\ln[A\exp(C+1/2)\equiv 2(\omega_1-\omega_0)$$

where $C = 0.577$ is the Euler constant, and the frequency ω_1 is defined using the above relationship $u_{\max}(\omega) = u_{\max}(\omega') \exp[-\alpha(\omega - \omega')]$:

$$\frac{4\pi\nu^2}{\delta^2} u_{\max}(\omega_1) \exp(2C + 1/2) = 1$$

or

$$\frac{\nu^2}{\delta^2} u_{\max}(\omega_1) = [4\pi \exp(2C + 1/2)]^{-1}$$

Let us now consider the other limiting case, when the line width ν is considerably greater than the mean frequency difference δ between the neighbouring transitions. Since the main contribution to the emission band width is made by the frequencies for which

$$\frac{\nu}{\delta} \int_{-\infty}^{+\infty} \left\{1 - \exp\left[-\frac{u_{\max}(\omega)}{(1+s^2)\cos\theta}\right]\right\} ds \sim 1$$

we can expand the exponent and obtain

$$\frac{\nu}{\delta} \int_{-\infty}^{+\infty} ds \left\{1 - \exp\left[-\frac{u_{\max}(\omega)}{(1+s^2)\cos\theta}\right]\right\} = \frac{\pi\nu u_{\max}(\omega)}{\delta\cos\theta}$$

This leads to the following expression for the emission band width:

$$\Delta\omega = 2\int_0^1 \cos\theta \, d(\cos\theta) \int_{-\infty}^{+\infty} d\omega \left\{1 - \exp\left[-\frac{\pi u_{\max}(\omega)}{\delta\cos\theta}\right]\right\}$$

As before, let us take a frequency ω' for which the exponent is still large but which is close to the frequency for which the exponent is of the order of unity. Approximating the rapidly varying function $u_{\max}(\omega)$ in this frequency range by $u_{\max}(\omega) = u_{\max}(\omega') \exp[-\alpha(\omega - \omega')]$, we obtain

$$\Delta\omega = 2(\omega' - \omega_0) + \frac{4}{\alpha}\int_0^1 \cos\theta \, d(\cos\theta) \int_0^{A/\cos\theta} \frac{dz}{z}[1 - \exp(-z)]$$

where ω_0 is the emission band centre, and

$$z = \frac{A}{\cos\theta}\exp[-\alpha(\omega - \omega')], \quad A = \frac{\pi\nu}{\delta} u_{\max}(\omega') \gg 1$$

As above, finding the asymptotic expression for the last integral for $A \gg 1$, we obtain

$$\Delta\omega = 2(\omega' - \omega_0) + \frac{4}{\alpha}\int_0^1 \cos\theta \, d(\cos\theta) \ln \frac{A \exp(C)}{\cos\theta}$$

$$= 2(\omega' - \omega_0) + \frac{2}{\alpha} \ln [A \exp(C + 1/2)] \equiv 2(\omega_1 - \omega_0)$$

where the frequency ω_1 is found from the relationship

$$\frac{v}{\delta} u_{\max}(\omega_1) = [\pi \exp(C + 1/2)]^{-1}$$

Problem 5.27. Determine the absorption coefficient of a molecular gas assuming that absorption is due to vibrational-rotational transitions in linear molecules. The spectral lines have the Lorentz shape and the line width is small compared with the frequency difference between the centres of neighbouring lines.

Assume that thermodynamic equilibrium between the vibrational-rotational states is maintained in the gas. Then the number density of molecules in the vibrational state v and the rotational state j is

$$N_{vj} = N_0 \exp(-\hbar\omega_0 v/T) \frac{B}{T}(2j + 1) \exp[-Bj(j + 1)/T]$$

where N_0 is the total number density of molecules in the ground vibrational state, $\hbar\omega_0$ is the excitation energy of the vibrational level, B is the rotational constant of the molecule, and T is the gas temperature in terms of energy units ($B \ll T$).

Using eq. (5.6) for the absorption coefficient and eq. (A4.2) for the line shape, we can find the coefficient of absorption due to excitation of molecules from the ground vibrational state:

$$k_\omega = \frac{\pi^2 c^2}{\omega^2} \sum_j w N_0 [1 - \exp(-\hbar\omega_0/T)]$$

$$\times \frac{B}{T}(2j + 1) \exp[-Bj(j + 1)/T] \frac{\nu}{\pi[(\omega - \omega_j)^2 + \nu^2]}$$

Here w is the frequency of spontaneous emission for the given transition, ν is the line width, $\hbar\omega_j$ is the transition energy and $\hbar\omega_j = \hbar\omega_0 + 2B(j + 1)$ for the transition in which the rotational quantum number is increased by unity (the P branch) and $\hbar\omega_j = \hbar\omega_0 - 2Bj$ for the transition in which the rotational quantum number is decreased by unity (the R branch).

Denote by τ the lifetime of the excited vibrational state of the molecule with respect to emission. Since $1/\tau = w(j \to j - 1) + w(j \to j + 1)$ and for large rotational quantum numbers ($j \gg 1$) we have $w(j \to j-1) = w(j \to j+1)$, then we obtain $w(j \to j-1) = w(j \to j+1) = 1/2\,\tau$. Taking this into account and assuming the rotational constant B to be small compared with the excitation energy $\hbar\omega_0$ and the thermal energy T, we can write the absorption coefficient as

$$k_\omega = \frac{\pi c^2 \nu}{\omega^2 \tau}\,\frac{B}{T}\,N_0[1-\exp(-\hbar\omega_0/T)]\sum_{j=-\infty}^{\infty}\frac{(j+1/2)\exp[-Bj(j+1)/T]}{\left[\omega-\omega_0-\frac{2B}{\hbar}(j+1/2)\right]^2+\nu^2}$$

Since $B \ll T$ and

$$\sum_{n=-\infty}^{\infty}\frac{1}{(x-n)^2+y^2} = \frac{\pi\sinh 2\pi y}{y(\cosh 2\pi y-\cos 2\pi x)}$$

we can rewrite the absorption coefficient in the following form:

$$k_\omega = \frac{\pi^2 c^2}{4\omega^2}\,\frac{\hbar^2|\omega-\omega_0|}{\tau TB}\,N_0[1-\exp(-\hbar\omega_0/T)]\times$$

$$\times\frac{\sinh\frac{\pi\hbar\nu}{B}\exp\left[-\frac{\hbar^2(\omega-\omega_0)^2}{4BT}\right]}{\left[\cosh\frac{\pi\hbar\nu}{B}-\cos\frac{\pi\hbar|\omega-\omega_0|}{B}\right]}$$

The absorption coefficient is an almost periodic function of frequency with the period $2B/\hbar$ ($2B$ is the energy difference between two neighbouring vibrational-rotational transitions). This dependence of the absorption coefficient is known as the *Elsasser function*. This function has a comb-like plot. The ratio between the absorption coefficients at the neighbouring maximum and minimum is $[1+\cosh(\pi\hbar\nu/B)]/[\cosh(\pi\hbar\nu/B-1]$. For instance, if $\hbar\nu \ll B$, this ratio is large and equal to $(2B/\pi\hbar\nu)^2$. The maximum absorption coefficient corresponds to the excitation of the rotational state in which there is the maximum number of molecules. The respective transition frequency is $|\omega-\omega_0| = \sqrt{2BT}/\hbar$ and the maximum absorption coefficient is

$$k_{\max} = \frac{\pi^2 c^2}{\omega^2}\,\frac{\hbar\sqrt{2}}{\tau\sqrt{eBT}}\,N_0[1-\exp(-\hbar\omega_0/T)]\frac{\sinh\frac{\pi\hbar\nu}{B}}{\left(\cosh\frac{\pi\hbar\nu}{B}-1\right)}$$

Problem 5.28. Find the emission band width for the quasiregular model and the Lorentz line shape. Assume that the line intensity sharply varies with the frequency in the frequency range comparable with the emission band width but the variation of the line intensity for the neighbouring lines is small.

According to the results of Problem 5.27, the optical thickness of the gas slab in this case is

$$u_\omega = u_m(\omega)\,\frac{\cosh\dfrac{2\pi\nu}{\delta} - 1}{\cosh\dfrac{2\pi\nu}{\delta} - \cos\dfrac{2\pi(\omega-\omega_0)}{\delta}}$$

where $u_m(\omega)$ is the optical thickness corresponding to the centre of the respective line, ν is the line width, $\delta = 2B/\hbar$ is the frequency difference between the neighbouring lines, and ω_0 is the frequency of the emission band centre. As before, we assume that the band is symmetric with respect to its centre. Moreover, the model assumes that the variation of $u_m(\omega)$ is small if the frequency variation is of the order of δ but at the band end the variation of $u_m(\omega)$ is noticeable for the frequency variations which are small compared with the emission band width.

According to the general equation (5.17), the emission band width is

$$\Delta\omega = 2\int_{-\infty}^{+\infty} d\omega \int_0^1 \cos\theta\, d(\cos\theta)$$

$$\times\left\{1 - \exp\left[-\frac{u_m(\omega)}{\cos\theta}\,\frac{\cosh\dfrac{2\pi\nu}{\delta} - 1}{\cosh\dfrac{2\pi\nu}{\delta} - \cos\dfrac{2\pi(\omega-\omega_0)}{\delta}}\right]\right\}$$

Let us consider separately two limiting cases. If the line width ν is large compared with δ, then we have

$$\Delta\omega = 2\int_0^1 \cos\theta\, d(\cos\ \theta) \int_{-\infty}^{+\infty} d\omega\left\{1 - \exp\left[-\frac{u_m(\omega)}{\cos\ \theta}\right]\right\}$$

Since $u_m(\omega)$ rapidly varies with frequency, we can calculate the integral using the procedure described in Problem 5.27. Namely, take ω' for which $u_m(\omega') \gg 1$ but which is close to ω_1 for which $u_m(\omega_1) \sim 1$. Then in the given frequency range we have $u_m(\omega) = u_m(\omega')\ \exp\ [-\alpha(\omega-\omega')]$, where α is the logarithmic derivative of the optical thickness $u_m(\omega)$ at the point ω'. Using these procedures, we obtain

$$\Delta\omega = 2(\omega' - \omega_0) + \frac{4}{\alpha}\int_0^1 \cos\ \theta\, d(\cos\ \theta) \int_0^{u_m(\omega')/\cos\theta} [1 - \exp(-z)]\frac{dz}{z}$$

where $z = u_m(\omega)/\cos\ \theta$.

Using the asymptotic expression for the last integral and taking into account the fact that $u_{\mathrm{m}}(\omega') \gg 1$, we find

$$\Delta\omega = 2(\omega' - \omega_0) + \frac{2}{\alpha} \ln\left[u(\omega') \exp\left(C + 1/2\right)\right] \equiv 2(\omega_1 - \omega_0)$$

where $u_{\mathrm{m}}(\omega_1) = \exp\left(-C - \frac{1}{2}\right)$ and $C = 0.577$ is the Euler constant.

In the other limiting case, when the line width ν is small compared with the frequency difference δ between the neighbouring lines, we can express the integral for $\Delta\omega$ as the sum of the integrals between the neighbouring lines and obtain

$$\Delta\omega = \frac{\delta}{\pi} \int_0^1 \cos\theta \, d(\cos\theta) \sum_n \int_0^{2\pi} d\varphi$$

$$\times \left\{1 - \exp\left[-\frac{u_{\mathrm{m}}(\omega)}{\cos\theta} \frac{\left(\cosh\frac{2\pi\nu}{\delta} - 1\right)}{\left(\cosh\frac{2\pi\nu}{\delta} - \cos\varphi\right)}\right]\right\}$$

where $2\pi(\omega - \omega_0)/\delta = \varphi + 2\pi n$. Assuming that the main contribution to the emission band width is made by many lines, we can replace summation over n by integration over frequency:

$$\Delta\omega = \frac{1}{\pi} \int_0^1 \cos\theta \, d(\cos\theta) \int_{-\infty}^{+\infty} d\omega \int_0^{2\pi} d\varphi$$

$$\times \left\{1 - \exp\left[-\frac{u_{\mathrm{m}}(\omega)}{\cos\theta} \frac{\left(\cosh\frac{2\pi\nu}{\delta} - 1\right)}{\left(\cosh\frac{2\pi\nu}{\delta} - \cos\varphi\right)}\right]\right\}$$

Here we have separated the oscillating part of the integral from the part which slowly varies during the period.

Expand now the integrand in powers of the small parameter ν/δ. The main contribution to the integral is made by the range of the variables where $\varphi \sim 1$, so that the hyperbolic cosine in the denominator can be replaced by unity. We obtain

$$\Delta\omega = \frac{1}{\pi} \int_0^1 \cos\theta \, d(\cos\theta) \int_{-\infty}^{+\infty} d\omega \int_0^{2\pi} d\varphi \left\{1 - \exp\left[-\frac{2\pi^2\nu^2 u_{\mathrm{m}}(\omega)}{\delta^2 \cos\theta\,(1 - \cos\varphi)}\right]\right\}$$

As before, take the frequency ω' for which $u_m(\omega') \gg 1$ but which is close to the frequency ω_1 satisfying the relationship $u_m(\omega_1) \sim 1$. Writing in this frequency range $u_m(\omega) = u_m(\omega') \exp[-\alpha(\omega - \omega')]$ and separating the integral over $d\omega$ into two parts, we obtain

$$\Delta\omega = 2(\omega' - \omega_0) + \frac{2}{\pi}\int_0^1 \cos\theta \, d(\cos\theta) \int_0^{2\pi} d\varphi \int_0^{\frac{A}{\cos\theta(1-\cos\varphi)}} \frac{dz}{z}[1 - \exp(-z)]$$

where

$$z = \frac{A\exp[-\alpha(\omega - \omega')]}{\cos\theta\,(1 - \cos\varphi)} \quad \text{and} \quad A = \frac{2\pi^2\nu^2}{\delta^2} u_m(\omega') \gg 1$$

Using the asymptotic expression for the last integral, we find

$$\Delta\omega = 2(\omega' - \omega_0) + \frac{2}{\pi\alpha}\int_0^1 \cos\theta \, d(\cos\theta)\int_0^{2\pi} d\varphi \ln \frac{A\exp(C)}{\cos\theta(1 - \cos\varphi)}$$

$$= 2(\omega' - \omega_0) + \frac{2}{\alpha}\ln[2A\exp(C + 1/2)] \equiv 2(\omega_1 - \omega_0)$$

where C is the Euler constant and the frequency ω_1 is given by

$$\frac{\nu^2}{\delta^2} u_m(\omega_1) = [4\pi^2 \exp(C + 1/2)]^{-1}$$

Let us relate the optical thickness corresponding to the centre of the respective transition to the optical thickness $u_{max}(\omega)$ corresponding to the line centre. For $\nu \ll \delta$ we have $u_m(\omega) = u_{max}(\omega)$, that is, the maximum optical thickness is due to one line. When $\nu \gg \delta$ and the maximum optical thickness is due to many transitions, we have $u_m(\omega) = u_{max}(\omega)\dfrac{\nu\pi}{\delta}$.

Taking this into account, we can transform the above result into

$$\Delta\omega = 2(\omega_1 - \omega_0)$$

When $\nu \gg \delta$, the frequency ω_1 is given by

$$\frac{\nu}{\delta} u_{max}(\omega_1) = [\pi\exp(C + 1/2)]^{-1}$$

and when $\nu \ll \delta$, this frequency is given by

$$\frac{\nu^2}{\delta^2} u_{max}(\omega_1) = [4\pi^2\exp(C + 1/2)]^{-1}$$

Problem 5.29. Find the mean absorption function for the given frequency and the effective width of the emission band for a slab of a gas, proceeding from the model of single lines and assuming the Lorentz line shape.

For the Lorentz line shape the optical thickness of the gas slab corresponding to one line is

$$u = u_{\max}\frac{\nu^2}{(\omega-\omega_i)^2+\nu^2} = \frac{u_{\max}}{1+s^2}$$

where $u_{\max}$ is the optical thickness corresponding to the line centre, ν is the line width, $\omega-\omega_i$ is the difference between the given frequency and the frequency corresponding to the line centre, and $s=(\omega-\omega_i)/\nu$.

In our case $u_{\max}\gg 1$ but the minimum optical thickness in the interval between two lines is much smaller than unity. Therefore, the integral in eq. (5.15) for the absorption function can be expressed as the sum of integrals taken about the centres of all lines where the effect of the neighbouring lines can be ignored. Hence, we obtain

$$A=\frac{\nu}{\delta}\int_{-\infty}^{+\infty}\left[1-\exp\left(-\frac{u_{\max}}{1+s^2}\right)\right]ds$$

where δ is the mean frequency difference between the neighbouring lines.

Since the main contribution to the integral is made by $s\gg 1$ $(u_{\max}\gg 1)$, we have

$$A=\frac{\nu}{\delta}\int_{-\infty}^{+\infty}\left[1-\exp\left(-\frac{u_{\max}}{s^2}\right)\right]ds=\frac{2\nu}{\delta}\sqrt{\pi u_{\max}}$$

so that $\bar{A}=\dfrac{2\nu\sqrt{\pi}}{\delta}\,\overline{(u_{\max})^{1/2}}$, where $\overline{[u_{\max}(\omega)]^{1/2}}$ is the mean square root of the optical thickness corresponding to the line with the centre at the given frequency.

According to eq. (5.18), the emission band width is

$$\Delta\omega=\nu\sum_i\int_0^1\cos\theta\, d(\cos\theta)\int_{-\infty}^{+\infty}ds\left\{1-\exp\left[-\frac{u^{(i)}_{\max}}{s^2\cos\theta}\right]\right\}$$

$$=\sum_i\int_0^1\cos\theta\, d(\cos\theta)\cdot 2\nu\sqrt{\pi u^{(i)}_{\max}\theta}=\frac{4\sqrt{\pi}\,\nu}{5}\sum_i\sqrt{u^{(i)}_{\max}}$$

where $u^{(i)}_{\max}$ is the optical thickness corresponding to the centre of the ith line. Assume that the emission band consists of a large number of individual lines. By definition, we have $u^{(i)}_{\max}=\dfrac{1}{2\pi\nu}\displaystyle\int S_i\,dx$, where S_i is the intensity of the ith line, and the integral is taken over the thickness dx of the gas slab. Assuming that the intensity distribution is specified for each frequency, we obtain

$$\Delta\omega = \frac{2\sqrt{2\nu}}{5}\int\sqrt{\int S(\omega)\,dx}\,p\,[S(\omega)]\,dS\,\frac{d\omega}{\delta}$$

If the line intensity is determined by the frequency, this equation is transformed into

$$\Delta\omega = \frac{2\sqrt{2\nu}}{5\delta}\int d\omega\sqrt{\int S(\omega)\,dx}$$

Problem 5.30. Calculate the absorption coefficient of a molecular gas, proceeding from the regular model and assuming the Lorentz line shape.

Using eq. (5.18) for the absorption coefficient and taking into account the Lorentz line shape, we find that for identical line intensities

$$k_\omega = \sum_{n=-\infty}^{\infty}\frac{S\nu}{\pi}\left[(\omega-\omega_0-n\delta)^2+\nu^2\right]^{-1}$$

where ν is the line width, ω_0 is the line centre taken as the band centre, and δ is the mean frequency difference between the lines. Using the Mittag-Leffler theorem for the sum over n,

$$\left[\sum_{n=-\infty}^{\infty}\left[(x-n)^2+y^2\right]^{-1} = \frac{\pi\sinh 2\pi y}{y\left(\cosh 2\pi y-\cos 2\pi x\right)}\right]$$

we obtain

$$k_\omega = \frac{S}{\delta}\sinh\frac{2\pi\nu}{\delta}\left[\cosh\frac{2\pi\nu}{\delta}-\cos\frac{2\pi(\omega-\omega_0)}{\delta}\right]^{-1}$$

The ratio of the maximum absorption coefficient to the minimum value is $\left(\cosh\frac{2\pi\nu}{\delta}+1\right)\Big/\left(\cosh\frac{2\pi\nu}{\delta}-1\right)$. If the line width is much smaller than the distance between the neighbouring lines, this ratio is $(\delta/\pi\nu)^2$.

Problem 5.31. Compare the distribution over optical thicknesses derived from the random model with the exact distribution, assuming that the regular model is applicable and the lines have the Lorentz shape.

The optical thickness of a gas slab, according to the regular model, is

$$u = u_{\min}\frac{\cosh\dfrac{2\pi\nu}{\delta}+1}{\cosh\dfrac{2\pi\nu}{\delta}-\cos\varphi}$$

where $u_{\min}$ is the optical thickness corresponding to the frequency ω halfway between two neighbouring lines, ν is the line width, δ is the frequency difference between two neighbouring lines, and $\varphi = \frac{2\pi}{\delta}(\omega - \omega_l)$ (here ω_l is the frequency at the centre of the nearest line).

The probability that the distance between the frequency ω taken at random and the centre of the nearest line is in the range from $\omega - \omega_l$ to $\omega + d\omega - \omega_l$ is proportional to $d\omega$ or $d\varphi$. Hence,

$$f(u)\,du = \frac{1}{\pi}\,d\varphi$$

and φ varies in the range from 0 to π. This leads to the distribution over optical thicknesses:

$$f(u) = \frac{1}{\pi}\left(\frac{du}{d\varphi}\right)^{-1}$$

Let us analyze the case when the line width ν is small compared with the frequency difference δ between the neighbouring lines. Then for the frequencies which are not too close to the line centre ($|\omega - \omega_l| \gg \nu$) we obtain

$$f(u)\,du = \frac{\sqrt{u_{\min}}\,du}{\pi u\sqrt{u - u_{\min}}}$$

In the random model we have derived the following distribution over optical thicknesses (see Problem 5.24):

$$f(u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt \exp\left\{itu - \int_{-\infty}^{+\infty}\frac{d\omega}{\delta}\left[1 - \exp\left(-\frac{itu_{\max}}{1+s^2}\right)\right]\right\}$$

Here $u_{\max}$ is the optical thickness corresponding to the centre of one of the transitions, $s = (\omega - \omega_l)/\delta$, ω is a given frequency, and ω_l is the frequency at the line centre. Using $u_{\min} = u_{\max}\left(\frac{\pi\nu}{\delta}\right)^2$ and, as before, considering the range of not too great optical thicknesses, we obtain

$$f(u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt \exp\left\{itu - \frac{1}{\pi}\int_{-\infty}^{+\infty} dy\left[1 - \exp\left(-\frac{itu_{\min}}{y^2}\right)\right]\right\}$$

where $y = \frac{\pi\nu}{\delta}\,s = \frac{\pi}{\delta}(\omega - \omega_l)$. Taking the integral over dy, we obtain

$$f(u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt \exp\left(itu - \sqrt{\frac{2}{\pi}tu_{\min}} - i\sqrt{\frac{2}{\pi}tu_{\min}}\right)$$

For $u \gg u_{\min}$ we have

$$f(u) = \frac{1}{\pi} \frac{u_{\min}^{1/2}}{u^{3/2}}$$

that is, this function coincides with the exact distribution. When $u \sim u_{\min}$, the distribution found from the random model differs from the exact distribution.

Problem 5.32. Find the absorption function and the emission band width for a slab of a molecular gas. The frequency distribution of the emitted photons sharply varies at the line wing. Employ the model of single lines assuming the Doppler line shape.

The absorption function in the model of single lines is

$$A = \int \frac{d\omega}{\delta} [1 - \exp(-u)]$$

For simplicity, assume that the line is symmetric with respect to its centre at which the frequency is ω_0. The above integral can be approximated by

$$A \approx \frac{2|\omega_0 - \omega_1|}{\delta}$$

where $u(\omega_1) \sim 1$ according to the form of the integral. To improve the approximation, take a frequency ω', such that $u(\omega') \gg 1$ but ω' is close to ω_1. Since the optical thickness rapidly varies with frequency in the frequency range about ω', we shall approximate the optical thickness by the function $u(\omega) = u(\omega') \exp[-\alpha(\omega - \omega')]$, where $\alpha = \left.\frac{d(\ln u)}{d\omega}\right|_{\omega'}$. Then, we obtain

$$A = \frac{2(\omega' - \omega_0)}{\delta} + \frac{2}{\delta} \int_{\omega'}^{\infty} d\omega \{1 - \exp[-u(\omega') \exp(-\alpha(\omega - \omega'))]\}$$

Using a new variable $z = u(\omega') \exp[-\alpha(\omega - \omega')]$, we obtain

$$A = \frac{2(\omega' - \omega_0)}{\delta} + \frac{2}{\delta\alpha} \int_0^{u(\omega')} \frac{dz}{z} [1 - \exp(-z)]$$

Since $u(\omega') \gg 1$, the asymptotic value of the integral is

$$\int_0^{u(\omega')} \frac{dz}{z} [1 - \exp(-z)] = \ln[u(\omega')] + C$$

where $C = 0.577$ is the Euler constant. Then the absorption function is

$$A = \frac{2(\omega' - \omega_0)}{\delta} + \frac{2}{\delta\alpha}\{\ln[u(\omega')] + C\} \equiv \frac{2(\omega_1 - \omega_0)}{\delta}$$

where $u(\omega_1) = \exp(-C)$.

The emission band width can be calculated in a similar way. The emission band is the sum of individual bands about the centre of each line, and we have to find their widths. We have

$$\Delta\omega = 2\int_0^1 \cos\theta\, d(\cos\theta) \int_{-\infty}^{+\infty} d\omega\,\{1 - \exp[-u(\omega)/\cos\theta]\}$$

where $u(\omega)$ steadily and sharply decreases with increasing distance from the line centre ω_0 and is symmetric with respect to the centre. Take the integral over $d(\cos\theta)$ by parts and obtain

$$\Delta\omega = \int_{-\infty}^{+\infty} d\omega\,[1 - \exp(-u)] + \int_{-\infty}^{+\infty} u\, d\omega \int_0^1 d(\cos\theta)\exp(-u/\cos\theta)$$

The first integral has been found before:

$$\int_{-\infty}^{+\infty} d\omega\,[1 - \exp(-u)] = 2(\omega_1 - \omega_0), \text{ where } u(\omega_1) = \exp(-C)$$

Using the above approximation $u(\omega) = u(\omega')\exp[-\alpha(\omega - \omega')]$, we can transform the second integral into

$$\int_{-\infty}^{+\infty} d\omega \int_0^1 d(\cos\theta)\, u\exp(-u/\cos\theta) = 2\int_0^{u_{\max}} \frac{du}{\alpha u}\int_0^1 d(\cos\theta)\exp(-u/\cos\theta) = \frac{1}{\alpha}$$

Finally, we have

$$\Delta\omega = 2(\omega_1 - \omega_0) + \frac{1}{\alpha} \equiv 2(\omega_2 - \omega_0), \quad \text{where} \quad u(\omega_2) = \exp(-C - 1/2).$$

For the Doppler line shape we have

$$u = u_{\max}\exp\left[-\frac{Mc^2}{2T}\,\frac{(\omega - \omega_0)^2}{\omega_0^2}\right]$$

where $u_{\max}$ is the optical thickness corresponding to the line centre ($u_{\max} \gg 1$), M is the molecular mass, T is the gas temperature, and c is the velocity of light. Finally, we obtain for the absorption function

$$A = \frac{2\omega_0}{\delta}\sqrt{\frac{T}{Mc^2}}\sqrt{(\ln u_{\max} + C)}$$

The emission band width is

$$\Delta\omega = 2\omega_0\sqrt{\frac{T}{Mc^2}}\sqrt{\left(\ln u_{\max} + C + \frac{1}{2}\right)}$$

Problem 5.33. The optical thickness of a molecular gas slab corresponding to a single line decreases as $(\omega - \omega_i)^{-n}$ with increasing distance from the line centre (here $n < 2$, ω_i is the frequency of the line centre, and $|\omega - \omega_i|$ is considerably greater than the line width). Find the emission band width $\Delta\omega$ using the random model and the regular model. Assume that the line width is considerably smaller than the mean frequency difference δ between the neighbouring lines, the emission band is symmetric with respect to the band centre ω_0, the line intensity sharply decreases towards the band edge, and the optical thickness halfway between two neighbouring lines is large.

According to the conditions of the problem, we can approximate the optical thickness of the gas slab for the frequencies ω which are not close to the line centre ω_i:

$$u = u_{\max} s^{-n}$$

Here $u_{\max}$ is the optical thickness corresponding to the line centre, and $s = |\omega - \omega_i|/\nu$, and ν is of the order of the line width (but not equal to it). For simplicity, assume that the line is symmetric with respect to its centre. Then the random model yields

$$\Delta\omega = 2\int_{-\infty}^{+\infty} d\omega \int_0^1 \cos\theta \, d(\cos\theta)$$

$$\times\left\{1 - \exp\left[-\frac{\nu}{\delta}\int_{-\infty}^{+\infty} ds(1 - \exp\{-u_{\max}/\cos\theta\, s^n\})\right]\right\}$$

The main contribution to the integral is made by the not too small values of s for which the above expansion is valid. Integration over ds yields

$$\Delta\omega = 4\int_0^1 \cos\theta \, d(\cos\theta) \int_0^\infty d\omega \left\{1 - \exp\left[-\frac{2\nu}{\delta}\left(\frac{u_{\max}}{\cos\theta}\right)^{1/n} \Gamma\left(1 - \frac{1}{n}\right)\right]\right\}$$

As before, we assume here that the optical thickness $u_{\max}(\omega)$ is symmetric with respect to the band centre.

Since $u_{\max}(\omega)$ rapidly varies in the frequency range of the above integral, we can make use of the approximation $u(\omega) = u(\omega') \exp[-\alpha(\omega - \omega')]$, where ω' is such that $u(\omega') \gg 1$ and α is the logarithmic derivative of the optical thickness at this point. Using new variables

$$z = A \exp[-\alpha(\omega - \omega')], \quad A = \frac{\nu}{\delta}[u_{\max}(\omega')]^{1/n}\Gamma\left(1 - \frac{1}{n}\right) \gg 1$$

and dividing the integral into two parts, we obtain

$$\Delta\omega = 2(\omega' - \omega) + \frac{4}{\alpha}\int_0^1 \cos\theta \, d(\cos\theta) \int_0^{\frac{A}{(\cos\theta)^{1/n}}} \frac{dz}{z}[1 - \exp(-z)]$$

Taking the asymptotic expression for the integral, we find

$$\Delta\omega = 2(\omega' - \omega) + \frac{2}{\alpha}\ln[A\exp(C)] + \frac{1}{\alpha n} \equiv 2(\omega_1 - \omega_0)$$

where ω_1 is found from

$$\left(\frac{\nu}{\delta}\right)^n u_{\max}(\omega_1) = \exp(-1/2)\left[2\exp(C)\Gamma\left(1 - \frac{1}{n}\right)\right]^{-1/n}$$

In the regular model the emission band width is

$$\Delta\omega = 2\int_0^1 \cos\theta \, d(\cos\theta) \int_{-\infty}^{+\infty} d\omega \left\{1 - \exp\left[-\frac{u_{\max}(\omega)}{\cos\theta} \sum_{k=-\infty}^{\infty} \frac{\nu^n}{|\omega - \omega_0 - \delta k|^n}\right]\right\}$$

Break the integral into parts using new variables $(\omega - \omega_0)/\delta = t + m$, where m is an integer and t continuously varies in the range $-1/2 \leqslant t \leqslant 1/2$.

Replacing summation over m with integration over $d\omega$, we obtain

$$\Delta\omega = 2\int_0^1 \cos\theta \, d(\cos\theta) \int_{-\infty}^{+\infty} d\omega \int_{-1/2}^{1/2} dt \left\{1 - \exp\left[-\frac{u_{\max}(\omega)}{\cos\theta} \sum_{k=-\infty}^{\infty} \frac{\nu^n}{\delta^n |t-k|^n}\right]\right\}$$

Since $u_{\max}(\omega)$ sharply varies with frequency, we can use the procedure described above and take $u_{\max}(\omega) = u_{\max}(\omega') \exp[-\alpha(\omega - \omega')]$. Using new variables $z = A \exp[-\alpha(\omega - \omega')]$ and $A = u_{\max}(\omega')\, \nu^n/\delta^n$, we obtain

$$\Delta\omega = 2(\omega' - \omega_0) + \frac{4}{\alpha} \int_{-1/2}^{1/2} dt \int_0^B \frac{dz}{z} [1 - \exp(-z)]$$

where

$$B = \frac{A}{\cos\theta} \sum_{k=-\infty}^{\infty} \frac{1}{|t-k|^n} \, .$$

Using the asymptotic expression for the integral over dz, we find

$$\Delta\omega = 2(\omega' - \omega_0) + \frac{2}{\alpha} \ln[A \exp(C + 1/2)]$$

$$+ \frac{2}{\alpha} \int_{-1/2}^{1/2} dt \ln\left[\sum_{k=-\infty}^{\infty} |t-k|^{-n}\right] \equiv 2(\omega_1 - \omega_0)$$

where

$$\frac{\nu^n}{\delta^n} u_{\max}(\omega_1) = \exp\left[-C - \frac{1}{2} - \int_{-1/2}^{1/2} dt \ln\left(\sum_{k=-\infty}^{\infty} |t-k|^n\right)\right]$$

Problem 5.34. The planet's surface absorbs the short-wave radiation of the Sun and converts it into long-wave radiation, behaving as the blackbody. The planet's atmosphere does not absorb the short-wave radiation. Analyze the effect of absorption of the long-wave radiation on the temperature of the planet's surface. Assume that the size of the atmosphere is small compared with the planet's diameter and the temperature of the atmosphere is equal to the temperature of the planet's surface.

As we have to find the mean radiation fluxes, we shall assume that the Sun's radiation is uniformly distributed over the planet's surface which converts it into the long-wave radiation. The atmosphere can absorb this radiation and emit long-wave radiation part of which gets to the planet's surface. Thus, the atmosphere increases the radiation flux incident on the planet's surface, raising its temperature.

Since the planet's diameter is large compared with the atmosphere thickness, the properties of the atmosphere at a given point are determined by the distance to the planet's surface. Denote the flux of the long-wave photons towards the planet's surface at a given height in the atmosphere by j_ω^-, the flux of long-wave photons from the planet's surface by j_ω^+, and the flux of photons of the given frequency by i_ω. Then eq. (5.11) yields

$$j_\omega^+(x) = \int_0^1 d(\cos\theta)\int_0^u du_\omega \exp[(-(u-u_\omega)/\cos\theta]j_\omega^{(0)} + i_\omega\int_0^1 d(\cos\theta)\exp(-u/\cos\theta)$$

$$j_\omega^-(x) = \int_0^1 d(\cos\theta)\int_0^u du_\omega \exp(-u_\omega/\cos\theta)\, j_\omega^{(0)}$$

where $u_\omega(x) = \int_0^x k_\omega\, dx'$ is the optical thickness of the atmospheric layer of the height x, k_ω is the absorption coefficient for the given frequency, x is the distance to the planet's surface, $u \equiv u_\omega(\infty) = \int_0^\infty k_\omega\, dx$ is the total optical thickness of the atmosphere, corresponding to the frequency ω, and $j_\omega^{(0)}(x)$ is the photon flux from the blackbody whose temperature is equal to the temperature of the given atmospheric layer. We should also take into account the fact that the total energy of the short-wave radiation absorbed by the planet's surface is equal to the total energy of the long-wave radiation escaping from the atmosphere. Thus we have

$$J_0 = \int \hbar\omega j_\omega^+(\infty)\, d\omega$$

The total energy radiated from the planet's surface is

$$J_s = J_0 + \int \hbar\omega j_\omega^-(0)\, d\omega$$

According to the Stefan-Boltzmann law, the temperature of the planet's surface emitting blackbody radiation is

$$T_s = (J_s/\sigma)^{1/4}$$

where σ is the Stefan-Boltzmann constant. If the planet's atmosphere freely transmits the long-wave radiation, that is, when $u \ll 1$, we have $j_\omega^- \ll j_\omega^{(0)}$. Then, according to the definition of $j_\omega^{(0)}$, we have $i_\omega = j_\omega^{(0)}$, so that $j_\omega^- \ll i_\omega$. Hence, $J_s = J_0$ and the temperature of the planet's surface is

$$T_s = (J_0/\sigma)^{1/4}$$

In the other limiting case, when the optical thickness of the atmosphere u is much greater than unity, we have $j_\omega^+(\infty) = j_\omega^-(0) = j_\omega^{(0)}$. If this relationship is satisfied for all frequencies, then we have $J_0 = \int \hbar\omega j_\omega^{(0)}\, d\omega$ and $J_s = J_0 + \int \hbar\omega j_\omega^{(0)}\, d\omega = 2J_0$. Hence, the temperature of the planet's surface is

$$T_s = (2J_0/\sigma)^{1/4}$$

Thus, if the atmosphere absorbs the radiation emitted from the planet's surface, the temperature of the surface is higher by a factor of $2^{1/4}$ (or by 19%) than the temperature in the absence of atmosphere. The atmosphere acts as a shield reflecting part of the radiation back to the planet's surface and thus increasing its temperature. This is illustrated by the diagram of the heat balance of a planet shown in Fig. 5.5. The balance of the radiative energy emitted and absorbed by the planet (the surface and atmosphere together) yields $\frac{1}{2} J_{\rm atm} = J_0 + I_0$, and the balance of the radiation

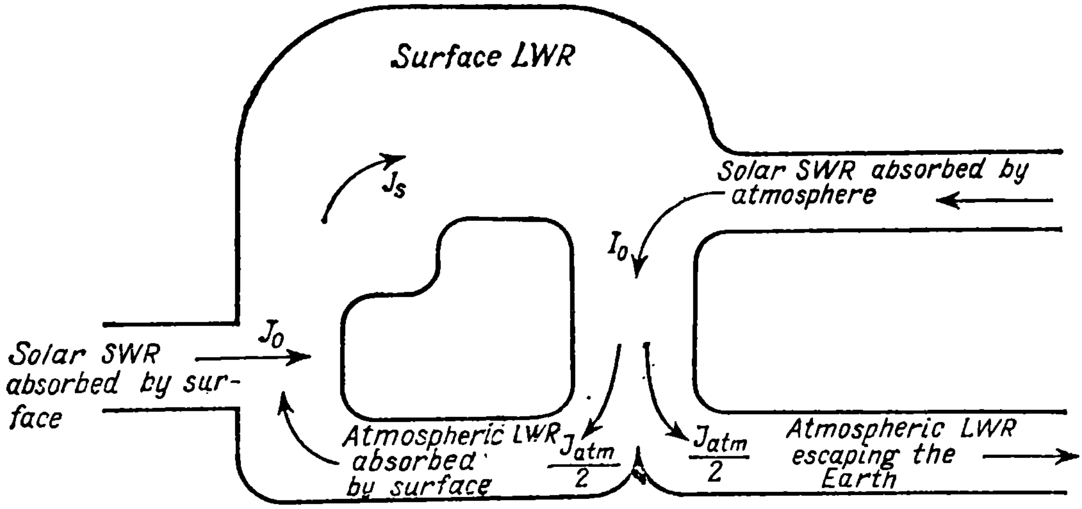
Surface LWR
J_s
Solar SWR absorbed by atmosphere
I_0
J_0
Solar SWR absorbed by surface
Atmospheric LWR absorbed by surface
$\frac{J_{atm}}{2}$
$\frac{J_{atm}}{2}$
Atmospheric LWR escaping the Earth

Fig. 5.5. The heat balance of the Earth. SWR — short-wave radiation. LWR — long-wave radiation.

energy absorbed and emitted by the atmosphere yields $J_{\text{atm}} = I_0 + J_s$. Eliminating J_{atm}, we find that the energy flux from the planet's surface is

$$J_s = 2J_0 + I_0$$

For instance, if the atmosphere fully transmits the short-wave radiation ($I_0 = 0$), we have $J_s = 2J_0$, that is, the radiation flux from the planet's surface is twice that obtained from the Sun or emitted in the absence of atmosphere. Note that under real conditions the temperature of the planet's surface is considerably higher than the temperature of the atmospheric layers from which the long-wave radiation escapes. Therefore, the long-wave radiation energy flux incident on the Earth's surface can be considerably higher than the solar radiation flux absorbed by the surface.

CHAPTER 6

SOME PROPERTIES OF THE GAS DISCHARGE

The processes analyzed above determine the parameters of a system with a weakly ionized gas. Systems of this type are illustrated in this chapter by the case of the gas discharge. The problems discussed here do not lay claim to full description of the physics of the phenomena and properties of gas discharges. Our goal is to demonstrate the possibilities of using information on various processes in plasma for the analysis of the characteristic of complex systems.

§ 6.1. THE DIFFUSION MODE OF THE POSITIVE DISCHARGE COLUMN

The positive column of the gas discharge is a typical example of a weakly ionized plasma. The electron number density in the positive columns of the glow discharge and the arc discharge is in the range from 10^8 to 10^{14} cm^{-3} and, since the mean electron energy is of the order of an electron-volt, the Debye-Hückel radius $r_D = \sqrt{T/8\pi Ne^2}$ for the positive column plasma is in the range from 10^{-5} to 10^{-2} cm. Thus, under normal laboratory conditions the Debye-Huckel radius is much smaller than the transverse dimensions of the system with a plasma, so that the plasma of the positive discharge column is quasineutral.

The processes described in the preceding chapters of this book occur in the plasma of the positive discharge column and its study yields interesting physical results. On the other hand, the study of the positive column plasma has a practical importance, since gas discharges are widely used.

The positive discharge column is assumed to be in the diffusion mode when the mean free path of electrons is small compared with the transverse dimensions of the discharge system and the escape of the charged particles to the walls is determined by the ambipolar diffusion. Moreover, the gas density in discharges of this type is not too high and the electron temperature differs considerably from the gas temperature.

Problem 6.1. Find the distribution of charged particles over the cross section of a discharge cylinder if charged particles are generated by direct ionization in the positive column and the disappearance of charged particles is determined by their escape to the walls.

The balance equation for the electron number density N_e in this case can be written as

$$\mathscr{D}_a \Delta N_e + k_{ion} N_a N_e = 0$$

where $\mathscr{D}_a$ is the ambipolar diffusion coefficient, k_{ion} is the rate constant for ionization of atoms in the ground state, and N_a is their number density. Owing to the cylindrical symmetry of the problem, we can reduce the balance equation to

$$\mathscr{D}_a \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dN_e}{d\rho}\right) + k_{ion} N_a N_e = 0 \tag{6.1}$$

where ρ is the distance from the electron to the axis of the discharge cylinder. The solution of this equation for the boundary condition $N_e(r_0) = 0$, where r_0 is the radius of the system, is

$$N_e = N_0 J_0(2.405\rho/r_0) \tag{6.2}$$

where N_0 is the electron number density at the axis of the system, and $J_0(x)$ is the Bessel function. The balance equation yields the following relationship between the plasma parameters and the size of the system:

$$N_0 k_{ion} r_0^2/\mathscr{D}_a = 5.78 \tag{6.3}$$

Equation (6.3) can be used to find the longitudinal electric field in the positive column, since the frequency of ionization is determined by the field strength. According to eq. (6.3), the frequency of generation of charged particles is equal to the frequency of the escape of charged particles to the walls.

Problem 6.2. Find the distribution of charged particles over the cross section of a cylindrical discharge tube, if charged particles are produced in a narrow region around the axis of the tube ($k_{ion} = k_0$ for $\rho \leqslant \rho_0$ and $k_{ion} = 0$ for $\rho > \rho_0$; $\rho_0 \ll r_0$).

This type of distribution of charged particles over the cross section of the tube can be due to the variation of the gas temperature with the distance from the tube axis. If the discharge current is not too small, the gas in the discharge tube is heated up owing to collisions between gas molecules and charged particles. The heat flux is directed towards the tube walls, so that the gas temperature at the axis is higher than at the walls. Since the gas pressure is spatially uniform, the number density of the gas molecules at the axis is lower than at the walls. Therefore, the collisions between charged particles and the gas molecules occur less frequently at the axis than at the walls and the mean energy of electrons at the axis is higher than that at the walls. Since ionization of the gas molecules is determined by the fast electrons at the tail of the distribution function, even a small difference in the gas temperature can result in a large difference between the ionization frequencies at the axis and at the walls.

Solving the balance equation (6.1) for the electron number density, we obtain

$$N_e = N_0\left(1 - \frac{k_{\text{ion}}N_a}{4\mathcal{D}_a}\rho^2\right) \quad \text{for} \quad \rho \leqslant \rho_0$$

$$N_e = N_1 \ln\left(\frac{r_0}{\rho}\right) \quad \text{for} \quad \rho_0 \leqslant \rho \leqslant r_0$$

Here we have used the boundary condition $N_e(r_0) = 0$. Let us join the above solutions at the point $\rho = \rho_0$, where the electron number density and its derivative (that is, the flux) must be continuous. We obtain

$$N_1 = N_0\left(1 - \frac{k_{\text{ion}}N_a}{4\mathcal{D}_a}\rho_0^2\right)$$

$$\frac{2\mathcal{D}_a}{k_{\text{ion}}N_a\rho_0^2} = \frac{1}{2} + \ln\left(\frac{r_0}{\rho_0}\right)$$

The last condition is physically equivalent to condition (6.3). It relates the frequency of generation of charged particles to the frequency of their escape from the volume where they are generated. Note that the relation includes the size of the region where electrons are generated rather than the entire region containing electrons. For $\rho_0 = r_0$ this condition reduces to eq. (6.3) in which 5.78 on the right-hand side is replaced by 4. This difference is explained by the fact that the assumption about smallness of the region with electrons in comparison with the discharge tube is not satisfied here.

To describe the degree of pinching of the discharge introduce the parameter

$$x = \frac{\int_0^{r_0} N_e \cdot 2\pi\rho \, d\rho}{N_e(0)\pi r_0^2}$$

which determines the fraction of the discharge tube volume occupied by electrons. If the distribution of electrons over the tube cross section is described by eq. (6.2), we have $x = 0.44$. For the above distribution we have

$$x = \frac{1 - 0.5y}{1 - \ln y}, \quad y = \frac{\rho_0^2}{r_0^2}$$

For instance, for $y = 1$ we obtain $x = 0.5$. The values of this function are given in the table below.

y	0.001	0.003	0.01	0.03	0.1	0.3	1
x	0.126	0.147	0.178	0.218	0.288	0.386	0.5

As we have seen, in this problem the region occupied by charged particles is considerably larger than the region in which they are generated. When we decrease here the region of electron generation by three orders of magnitude, the volume of the region containing electrons is changed by the factor of four. Therefore, compression of the region of the electron generation does not lead to pinching of the discharge.

Problem 6.3. A discharge is maintained between two infinite plates. Find the distribution of electrons in the discharge gap and formulate the condition for the electric field in the positive discharge column, if the recombination coefficient α for charged particles at the walls differs from unity. Assume that the frequency of ionization of atoms does not vary over the discharge cross section.

This problem with a simple geometry demonstrates the effect of the condition at the walls on the distribution of charged particles in the discharge gap. Write down the balance equation which is similar to eq. (6.1):

$$\mathscr{D}_a \frac{d^2 N_e}{dx^2} + k_{ion} N_e N_a = 0$$

Here x is the distance from the central plane of symmetry parallel to the walls of the discharge gap. The wall coordinates are $x = \pm\, l/2$, where l is the distance between the walls.

Taking into account the symmetry of the problem $[N_e(x) = N_e(-x)]$, we obtain the following solution of the above equation:

$$N_e = N_0 \cos \sqrt{\frac{k_{ion} N_a}{\mathscr{D}_a}}\, x$$

Here N_0 is the electron number density at the symmetry plane. Now we have to find the condition for the argument of the above function. In the limiting case, when the probability of recombination of charged particles at the walls is unity, the boundary condition $N_e(l/2) = 0$ yields

$$\sqrt{\frac{k_{ion} N_a}{\mathscr{D}_a}} \frac{l}{2} = \frac{\pi}{2}$$

Physically, this relation is equivalent to eq. (6.3); it relates the frequency of generation of charged particles in the volume to the frequency of their escape to the walls.

The flux of charged particles recombining at the walls is $\frac{N_e \bar{v}}{4}\alpha$, where $\bar{v}$ is the mean velocity of ions near the walls in the layer with the thickness of the order of the mean free path of ions, which is assumed to be much greater than the width of the Langmuir layer where the charge separation occurs. On the other hand, since ionization in the vicinity of the walls is insignificant, the flux of electrons recombining at the walls is $-\mathscr{D}_a \frac{dN(l/2)}{dx}$. Hence, the boundary condition at the walls is

$$\frac{N_e \bar{v}}{4}\alpha = -\mathscr{D}_a \frac{dN_e}{dx}$$

Using the above equation for the electron number density, we obtain

$$\cot \sqrt{\frac{k_{\text{ion}} N_a}{\mathscr{D}_a}} \frac{l}{2} = \frac{4\sqrt{k_{\text{ion}} N_a \mathscr{D}_a}}{\bar{v}\alpha}$$

Let us estimate the right-hand side of this equation. We have $\mathscr{D}_a \sim \bar{v}^2/\nu_{el}$, where ν_{el} is the frequency of collisions between ions and the gas atoms. Hence, the order-of-magnitude estimate for the right-hand side is $\frac{1}{\alpha}\sqrt{\nu_{\text{ion}}/\nu_{el}}$, where $\nu_{\text{ion}} = k_{\text{ion}} N_a$ is the frequency of ionization of the gas atoms by electrons. Since $\nu_{\text{ion}} \ll \nu_{el}$, then for $\alpha \sim 1$ the right-hand side is much smaller than unity, and we can write

$$\sqrt{\frac{k_{\text{ion}} N_a}{\mathscr{D}_a}} \frac{l}{2} = \frac{\pi}{2} - \frac{4\sqrt{k_{\text{ion}} N_a \mathscr{D}_a}}{\bar{v}\alpha}$$

Hence, we obtain

$$\sqrt{\frac{k_{\text{ion}} N_a}{\mathscr{D}_a}} \frac{l}{2} = \frac{\frac{\pi}{2}}{1 + \frac{8\mathscr{D}_a}{\bar{v} l \alpha}}$$

Let us estimate the second term in the denominator of the right-hand side. We have $\mathscr{D}_a \sim \bar{v}\lambda$, where λ is the mean free path of ions. Hence, this term is of the order of $\lambda/l\alpha$ and when

$$\alpha \gg \lambda/l$$

the boundary condition is the same as in the case when the electron number density at the wall is assumed to be zero. Thus, when the last condition is satisfied, the distribution of electrons in the discharge gap and the relationship between the frequency of generation of charged particles and the frequency of their escape to the walls are independent of the probability of recombination of charged particles at the walls.

Problem 6.4. Find the mean lifetime of a charged particle in a cylindrical discharge tube.

Charged particles disappear owing to their recombination at the walls of the discharge tube, so that their lifetime in the discharge is determined by the time of their escape to the walls, estimated by $\tau \sim r_0^2/\mathscr{D}_a$, where r_0 is the discharge tube radius, and $\mathscr{D}_a$ is the coefficient of ambipolar diffusion of charged particles. To improve the estimate make use of the distribution of charged particles over the tube radius:

$$N_e = N_0 J_0\left(2.4\frac{\rho}{r_0}\right)$$

Here N_0 is the number density of charged particles at the tube axis, and ρ is the distance from a given point to the axis. Hence, the flux of charged particles at the discharge tube walls is

$$j = -\mathscr{D}_a \frac{dN_e}{d\rho}\bigg|_{r_0} = -\frac{2.4\mathscr{D}_a N_0}{r_0}\frac{dJ_0}{dx}\bigg|_{x=2.4} = \frac{1.25\mathscr{D}_a N_0}{r_0}$$

The total number of charged particles in the tube element with the length dl is

$$dl\int_0^{r_0} 2\pi\rho\, d\rho\, N_e(\rho) = 2\pi r_0^2\, dl\, N_0 \int_0^1 x\, dx\, J_0(2.4x) = \frac{2\pi r_0^2\, dl\, N_0 J_1(2.4)}{2.4}$$

The number of charged particles per unit time from this tube element is $2\pi r_0 dlj$. Hence, the mean time τ spent by a charged particle in the discharge tube can be found as the ratio of these quantities:

$$\tau = \frac{\int_0^{r_0} 2\pi\rho\, d\rho\, N_e\, dl}{2\pi r_0\, dlj} = \frac{0.173 r_0^2}{\mathscr{D}_a}$$

We see here the same dependence on the problem parameters as in the initial estimate; the above calculations have just given the numerical factor which proves to be equal to that in eq. (6.3) which relates the time of production $(k_{ion}N_a)^{-1}$ and the time τ of disappearance for charged particles.

Problem 6.5. Estimate the size of the region occupied by charged particles if ionization occurs in a narrow region around the axis of the discharge tube and charged particles disappear owing to bulk recombination.

As shown in Problem 6.2, ionization in a narrow region does not lead to pinching of the discharge if charged particles disappear at the walls. Things are different in the case of bulk recombination.

Let us estimate the width of the region occupied by charged particles assuming that it is greater than the width of the region in which charged particles are produced. The time of recombination of a charged particle with an oppositely charged particle is $\tau \sim 1/\alpha N_e$, where α is the recombination coefficient. In this time the charged particle can travel the distance $r \sim \sqrt{\mathscr{D}_a \tau} \sim \sqrt{\mathscr{D}_a/\alpha N_e}$ from the tube axis. If this distance is much smaller than the tube radius, then it describes the size of the discharge region.

Let us estimate this size for real discharge conditions. Take the gas pressure of the order of 1 torr, so that the ambipolar diffusion coefficient $\mathscr{D}_a \sim 10^3$ cm^2/s, the two-particle recombination coefficient $\alpha \sim 10^{-7}$ cm^3/s, and the electron number density $N_e \sim 10^{12}$ cm^{-3}. Then the width of the discharge region is $r \sim 0.1$ cm, that is, the positive column of the discharge contracts under these conditions.

Problem 6.6. Find the distribution of electrons over the cross section of the positive discharge column in a cylindrical discharge tube after the discharge has finished.

The balance equation for the electron number density N_e is

$$\frac{\partial N_e}{\partial t} = \mathscr{D}_a \Delta N_e + \nu_{\text{ion}} N_e$$

where the ionization frequency ν_{ion} is determined by the form of the electron velocity distribution. Let ν_{ee} be the frequency of variation of the electron energy, so that $1/\nu_{ee}$ is the characteristic time of electron energy variation due to collisions with atoms. For instance, if electrons lose energy in elastic collisions with atoms, then $\nu_{ee} \sim m\nu_{el}/M$, where ν_{el} is the frequency of elastic collisions between electrons and atoms, m is the electron mass, and M is the atomic mass. We shall consider the limiting cases $\nu_{ee}\tau \ll 1$ and $\nu_{ee}\tau \gg 1$, where $\tau \sim r_0^2/\mathscr{D}_a$ is the characteristic time of escape to the walls for electrons, and r_0 is the radius of the discharge tube.

When $\nu_{ee}\tau \gg 1$, the mean electron energy varies much faster than the electron number density, that is, in a short time, ν_{ion} practically vanishes and the balance equation takes the form

$$\frac{\partial N_e}{\partial t} = \mathscr{D}_a \Delta N_e$$

The solution of this equation is

$$N_e = N_0 \exp(-t/\tau) J_0 \left(2.4 \frac{\rho}{r_0}\right), \quad \text{where} \quad \tau = r_0^2/5.76\mathscr{D}_a$$

When $\nu_{ee}\tau \ll 1$, the equation has the same solution but with

$$\tau(t) = r_0^2/5.76\mathscr{D}_a - 1/\nu(t)$$

Problem 6.7. Find the potential difference between the axis and the walls of a discharge tube for the positive column of gas discharge. The tube radius r_0 is considerably greater than the mean free path λ of electrons, the electron number density is sufficiently high for the electron velocity distribution to have the Maxwellian form, and the electron temperature is T_e.

The plasma of the positive gas discharge column is quasineutral, so that the radial currents of electrons and ions are identical. The electron current density is

$$\mathbf{j}_e = -\mathscr{D}_e \nabla N_e + \mathbf{w} N_e$$

where N_e is the electron number density, $\mathscr{D}_e$ is the diffusion coefficient for electrons, and $\mathbf{w}$ is the drift velocity of electrons in the radial direction.

Since the electron current is equal to the ion current, we can assume the electron current to be zero, taking into account the difference in masses between electrons and atoms. Then we obtain the following equation for the electron number density:

$$\mathscr{D}_e \nabla N_e = \mathbf{w} N_e$$

The solution of this equation is

$$N_e = N_0 \exp\left(-\int_0^\rho \frac{w}{\mathscr{D}_e} d\rho'\right)$$

where N_0 is the electron number density at the tube axis, and ρ is the distance from the given point to the tube axis.

Assume that the electron drift velocity w is small compared with the thermal velocity. Then we have $\mathbf{w} = K\mathbf{E}$, where $\mathbf{E}$ is the self-consistent field of electrons and ions, K is the electron mobility, and according to the Einstein relation, $K = e\mathscr{D}_e/T_e$. Since $\mathbf{E} = -\nabla U(\rho)$, where U is the electric field potential, we now obtain

$$N_e = N_0 \exp(eU/T_e)$$

where $U(\rho) \leqslant 0$ and at the tube axis $U = 0$.

Thus, we have found that the distribution of ions and electrons in the cross section of the discharge tube has the Boltzmann form. This distribution should be compared with the distribution (6.2) which was derived under the assumption that the ionization frequency is independent of the cross section of the discharge tube. Thus, we can find the potential at a given point:

$$eU = T_e \ln [J_0(2.4\rho/r_0)]$$

Let us demonstrate the validity of the above assumption about the smallness of the electron drift velocity in comparison with the thermal velocity. This is true if the energy $eE\lambda$ acquired by the electron along the mean free path is small compared with its thermal energy T_e. Let us estimate the parameter $eE\lambda/T_e$. We have

$$eE/T_e = e|\nabla U|/T_e = |\nabla N_e|/N_e \sim 1/r_0$$

if the electron is not too close to the wall, so that $eE\lambda/T_e \sim \lambda/r_0 \ll 1$, that is, the above condition is satisfied.

The above equations for the electron current and the electron number density are valid if the distance from the given point to the discharge tube wall is considerably greater than the mean free path of electrons. Hence, we find that the potential difference between the axis and the walls of the discharge tube is given by

$$eU = -T_e \ln (r_0/\lambda) \qquad (6.4)$$

Equation (6.4) to a constant factor of the order of unity in the logarithm yields the energy lost by the electron or acquired by the ion in motion from the axis to the walls of the discharge tube.

Problem 6.8. Find the degree of ionization of the gas discharge plasma in the positive discharge column, when escape of the excited atoms to the walls does not affect the coefficient of stepwise ionization of atoms. The electron temperature T_e is high compared with the gas temperature T but small compared with the excitation potential of the atom. The electron number density is high enough to neglect radiation by the excited atoms.

The stepwise ionization mechanism is effective when condition (4.40) is satisfied. However, the escape of the excited atoms to the discharge tube walls does not affect the coefficient of stepwise ionization if a more strict condition is met, namely, that the time of ionization of the excited atoms is shorter than the time of their escape to the walls. This condition can be written as

$$k_{\text{ion}}^{*} N_e \gg \mathscr{D}_{\text{exc}}/r_0^2$$

where k_{ion}^{*} is the rate constant of stepwise ionization of the excited atoms given by eq. (4.43), N_e is the electron number density, $\mathscr{D}_{\text{exc}}$ is the diffusion coefficient of the excited atoms in the parent gas, r_0 is the discharge tube radius, and $\mathscr{D}_{\mathrm{exc}}/r_0^2$ is the frequency of escape of the excited atoms to the walls.

Let us make use of the relationship between the parameters of the plasma of the positive discharge column. The frequency of production of free electrons is equal to the frequency of their escape to the tube walls, so that we have

$$k_{\mathrm{ion}}N_{\mathrm{a}} \sim \mathscr{D}_{\mathrm{a}}/r_0^2$$

where N_{a} is the number density of atoms in the ground state, k_{ion} is the rate constant of stepwise ionization of atoms in the ground state, and $\mathscr{D}_{\mathrm{a}}/r_0^2$ is the frequency of escape of electrons to the walls. The ambipolar diffusion coefficient is $\mathscr{D}_{\mathrm{a}} = \frac{T_{\mathrm{e}}}{T}\mathscr{D}_{\mathrm{i}}$, where $\mathscr{D}_{\mathrm{i}}$ is the diffusion coefficient for ions, T_{e} is the electron temperature, and T is the ion temperature which we assume to be equal to the gas temperature.

The diffusion coefficient $\mathscr{D}_{\mathrm{exc}}$ for the atoms in the lower excited states and the diffusion coefficient $\mathscr{D}_{\mathrm{i}}$ for ions are of the same order of magnitude because the cross sections of collisions of these particles with the parent atoms are of the same order. Hence, we obtain from the above expressions

$$\frac{N_{\mathrm{e}}}{N_{\mathrm{a}}} \gg \frac{T}{T_{\mathrm{e}}}\frac{k_{\mathrm{ion}}}{k_{\mathrm{ion}}^{*}}$$

Since the stepwise ionization proceeds via excitation of atoms in the lower state, we have $N_{\mathrm{a}}k_{\mathrm{ion}} = N_{\mathrm{exc}}k_{\mathrm{ion}}^{*}$, where N_{exc} is the number density of atoms in the lower excited state, and we obtain

$$N_{\mathrm{e}} \gg \frac{T}{T_{\mathrm{e}}}N_{\mathrm{exc}}$$

that is, stepwise ionization occurs if the number density of ions is higher than the number density of the excited atoms. If this condition is satisfied, the escape of the excited atoms to the walls does not affect the rate constant of stepwise ionization. Assuming that the distribution of atoms over the lower excited states has the Boltzmann form, we can rewrite the above relation as

$$\frac{N_{\mathrm{e}}}{N_{\mathrm{a}}} \gg \frac{T}{T_{\mathrm{e}}}\frac{g_{\mathrm{exc}}}{g_0}\exp\left(-\Delta E/T_{\mathrm{e}}\right)$$

where ΔE is the excitation energy, g_{exc} is the statistical weight of the lower excited state, and g_0 is the statistical weight of the ground state. This condition can be derived directly from the first condition for the degree of ionization of plasma, using eq. (4.43) for the ionization rate constants for atoms in the ground and excited states.

Problem 6.9. Find the distribution of charged particles in the cross section of the positive discharge column in a gas discharge tube. Ionization has the stepwise mechanism and occurs mainly via excitation of a resonance atomic state which disappears owing to radiation.

The balance equations for the electron number density N_e and the number density N_{exc} of the excited atoms are

$$\begin{aligned} \frac{dN_e}{dt} &= 0 = \mathcal{D}_a \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dN_e}{d\rho}\right) + k_{ion}N_eN_{exc} \\ \frac{dN_{exc}}{dt} &= 0 = N_eN_0k_{exc} - \frac{N_{exc}}{\tau} \end{aligned} \tag{6.5}$$

where ρ is the distance from the discharge tube axis, $\mathcal{D}_a$ is the ambipolar diffusion coefficient, k_{ion} is the ionization rate constant, k_{exc} is the rate constant for excitation of the atom from the ground state to the excited resonance state, N_0 is the number density of the gas particles, and $1/\tau$ is the probability of photon emission per unit time for the excited atom. This system of equations has to be solved with the boundary conditions

$$N_e(r_0) = N_{exc}(r_0) = 0, \quad \left.\frac{dN_e}{d\rho}\right|_{\rho=0} = \left.\frac{dN_{exc}}{d\rho}\right|_{\rho=0} = 0$$

where r_0 is the discharge tube radius. We shall take the number densities of electrons and excited atoms in the following approximated form:

$$N_e = C_1[\exp(-\delta\rho^2) - \exp(-\delta r_0^2)], \quad N_{exc} = C_2[\exp(-\gamma\rho^2) - \exp(-\gamma r_0^2)]$$

The parameters C_1, C_2, γ and δ can be found from the above balance equations for $\rho = 0$ and from the balance equation in the tube cross section integrated over $\rho\, d\rho$.

To verify this technique let us apply it to eq. (6.1). The resulting solution is eq. (6.2) and the parameters are related by eq. (6.3). Now, this technique yields $\delta r_0^2 = 0.842$, and instead of eq. (6.3) we obtain

$$\frac{v r_0^2}{\mathcal{D}_a} = \frac{4\delta r_0^2}{1 - \exp(-\delta r_0^2)} = 5.9$$

In order to compare the resulting distribution of electrons in the tube cross section and eq. (6.2), we have to calculate the integral

$$I = \int_0^{r_0} \left[C_1[\exp(-\delta\rho^2) - \exp(-\delta r_0^2)] - J_0\left(2.405\frac{\rho}{r_0}\right)\right]^2 \rho\, d\rho$$

for which we have

$$C_1\int_0^{r_0}[\exp(-\delta\rho^2)-\exp(-\delta r_0^2)]\,\rho\,d\rho=\int_0^{r_0}J_0\left(2.405\frac{\rho}{r_0}\right)\rho\,d\rho$$

We obtain $I=2.26\cdot 10^{-7}$, that is, above solution practically coincides with eq. (6.2).

Applying this technique to the balance equations (6.5), we obtain $N_{exc}=N_eN_0k_{exc}\tau$, that is, $\gamma=\delta$. Furthermore, $\delta r_0^2=1.5$ and the condition relating the parameters of the discharge plasma is

$$\frac{N_e(0)\,k_{ion}N_0k_{exc}\tau}{\mathcal{D}_a}\,r_0^2=\frac{4\delta r_0^2}{1-\exp(-\delta r_0^2)}=7.73 \tag{6.6}$$

As eq. (6.2), this condition yields an equation for the electric field of the gas discharge. Since the number density of excited atoms $N_{exc}\sim N_eN_0k_{exc}\tau$, this condition, as eq. (6.2), means that the frequency of production of charged particles, which is $\sim N_{exc}k_{ion}$, is equal to the frequency of their escape to the walls of the tube, which is $\sim \mathcal{D}_a/r_0^2$.

Problem 6.10. The conditions are the same as in Problem 6.9 but stepwise ionization proceeds via the excitation of metastable states which disappear owing to their escape to the discharge tube walls.

The balance equations for the electron number density N_e and the number density N_m of the metastable excited atoms are

$$\begin{aligned}&\mathcal{D}_a\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dN_e}{d\rho}\right)+k_{ion}N_mN_e=0\\&\mathcal{D}_m\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dN_m}{d\rho}\right)+k_{exc}N_0N_e=0\end{aligned} \tag{6.7}$$

and the boundary conditions are

$$N_e(r_0)=N_m(r_0)=0,\quad \left.\frac{dN_e}{d\rho}\right|_{\rho=0}=\left.\frac{dN_m}{d\rho}\right|_{\rho=0}=0$$

Here k_{ion} is the rate constant for ionization of the metastable atoms, $k_{exc}N_0$ is the frequency of production of the metastable states, and $\mathcal{D}_m$ is the diffusion coefficient for the metastable atoms. Using the procedure described in Problem 6.9, we obtain

$$N_e=C_1[\exp(-\delta\rho^2)-\exp(-\delta r_0^2)],\quad N_m=C_2[\exp(-\gamma\rho^2)-\exp(-\gamma r_0^2)]$$

where $\delta r_0^2=1.435$, $\gamma r_0^2=0.988$,

$$\frac{r_0^2k_{ion}N_m(0)}{\mathcal{D}_a}=7.55,\quad \frac{r_0^2k_{exc}N_0N_e(0)}{N_m(0)\,\mathcal{D}_m}=6.28 \tag{6.8}$$

According to the first of the above expressions, the frequency of production of charged particles is equal to the frequency of their escape to the walls. The second condition relates the number densities of electrons and metastable atoms.

Problem 6.11. Consider the similarity properties of the positive column of the gas discharge in a cylindrical gas discharge tube.

In the case of direct ionization of atoms, the relationship between discharge parameters is described by eq. (6.3). The kinetic equation for the electron distribution function shows that the ionization frequency $\nu_{\text{ion}} = N_a f(E/N_a)$, where N_a is the number density of atoms, and the form of the function $f(x)$ depends only on the gas properties. Since the ambipolar diffusion coefficient $\mathcal{D}_a \sim 1/N_a$, then eq. (6.3) shows that for similar discharges

$$N_a r_0 = \text{const} \quad \text{and} \quad E r_0 = \text{const} \tag{6.9}$$

where r_0 is the discharge tube radius, and E is the electric field. Since eq. (6.2) is linear, the similarity here is conserved for any electron number density, that is, for any current density.

In the case of stepwise ionization the electron number density cannot have an arbitrary value, since the balance equations for the electron number density are not linear. Equations (6.5) and (6.7) yield the following similarity relations in this case:

$$N_a r_0 = \text{const}, \quad E r_0 = \text{const}, \quad \text{and} \quad N_e r_0 = \text{const} \tag{6.10}$$

Problem 6.12. A readily ionizable admixture with the initial number density N_{ad} is added to a gas. The charged particles are produced in the positive discharge column only owing to the ionization of the admixture atoms. Find the distribution of electrons and admixture atoms over the discharge tube cross section if the diffusion coefficient of the admixture atoms is $\mathcal{D}$, the ambipolar diffusion coefficient is $\mathcal{D}_a$, the rate constant for electron impact ionization of the admixture atoms at the given conditions is k_{ion}.

The balance equations for the number densities of the admixture atoms N_a and electrons N_e during the discharge and the boundary conditions for them are

$$\begin{gathered}
\frac{\mathcal{D}}{\rho}\frac{d}{d\rho}\left(\rho\frac{dN_a}{d\rho}\right) - N_e N_a k_{\text{ion}} = 0 \\
\frac{\mathcal{D}_a}{\rho}\frac{d}{d\rho}\left(\rho\frac{dN_e}{d\rho}\right) + N_e N_a k_{\text{ion}} = 0 \\
N_e(R_0) = 0, \quad \int_0^{R_0} [N_e(\rho) + N_a(\rho)]\,\rho\, d\rho = N_{ad}\frac{R_0^2}{2} \\
\left.\frac{dN_e}{d\rho}\right|_{\rho=0} = \left.\frac{dN_a}{d\rho}\right|_{\rho=0} = 0
\end{gathered} \tag{6.11}$$

Rewrite the system of the balance equations in the dimensionless variables $x = \rho^2/R_0^2$, $n_e = N_e/N_{ad}$, $n_a = N_a/N_{ad}$, $a = 4\mathscr{D}/R_0^2 k_{ion} N_{ad}$, and $b = 4\mathscr{D}_a/R_0^2 k_{ion} N_{ad}$:

$$\left.\begin{aligned} a\frac{d}{dx}\left(x\frac{dn_a}{dx}\right) - n_e n_a = 0 \\ b\frac{d}{dx}\left(x\frac{dn_e}{dx}\right) + n_e n_a = 0 \end{aligned}\right| \int_0^1 (n_e + n_a)\,dx = 1$$

Adding up the above two equations and taking into account the fact that only recombination but not absorption of particles occurs at the tube walls, we obtain

$$a\frac{dn_a}{dx} + b\frac{dn_e}{dx} = 0$$

As before, take

$$n_a = C_1 - C_2 \exp(-\delta x), \quad n_e = C_3[\exp(-\delta x) - \exp(-\delta)]$$

where $C_3 = \frac{a}{b} C_2$ and the condition $\int_0^1 (n_e + n_a)\,dx = 1$ yields

$$C_2\left[\left(\frac{a}{b} - 1\right)\left(\frac{1 - \exp(-\delta)}{\delta}\right) - \frac{a}{b}\exp(-\delta)\right] + C_1 = 1$$

The balance equation for the electron number density at the tube axis yields

$$C_1 - C_2 = b\frac{\delta}{1 - \exp(-\delta)}$$

Finally, integrating the balance equation for electrons in the discharge tube cross section, we obtain

$$\delta\exp(-\delta) = C_1\left[\frac{1 - \exp(-\delta)}{\delta} - \exp(-\delta)\right] - \frac{C_2}{2\delta}[1 - \exp(-\delta)]^2$$

From the above three expressions we find the parameters C_1, C_2, and δ needed to express the number densities of electrons and atoms. Furthermore, we can find the range of variation of the parameters a and b.

Problem 6.13. An admixture of atoms with the number density N_{ad} is added to a gas in the positive column of a gas discharge. The concentration of the admixture is small, so that the collisions between charged particles and the gas atoms do not affect the properties of the positive column. Find the distribution of the admixture atoms along the column, assuming that the mass M of the admixture atoms is much larger than the mass m of the gas atoms and the ions are produced in the discharge only from the admixture atoms.

Each charged particle acquires per unit time the momentum eE in the electric field E and the direction of this momentum is determined by the charge sign. This momentum is transferred to the gas particles, so that the total momentum transferred to the gas is zero owing to the quasineutrality of plasma. In a mixture of gases the momentum transferred by charged particles to a given gas component can differ from zero, and thus the gas components can be separated under the effect of the discharge current.

Let the momentum **f** be transferred per unit time to each particle of a given species. According to the Boltzmann law, this results in the following distribution of the atoms of this species along the positive discharge column:

$$N = N_0 \exp(-fx/T)$$

Here T is the temperature, N_0 is the maximum number density of this species, and x is the coordinate along the discharge tube.

Let us find the force acting on the admixture atom under the effect of the current of charged particles. The momentum transferred to the gas atoms by a charged particle is given by the right-hand side of eq. (1.22). For instance, the momentum transferred by electrons to the admixture atoms is

$$m_e \int f_e(\mathbf{v})\, v_x \nu_{e\,ad}\, d\mathbf{v}$$

where the x-axis is parallel to the electric field, and $\nu_{e\,ad}$ is the frequency of collisions between electrons and the admixture atoms. If we make use of eqs. (2.20) and (2.24) for the velocity distribution of electrons in the uniform and constant electric field, we find that the force with which an electron acts on the admixture atoms is

$$\frac{eE}{3}\left\langle \frac{1}{v^2}\frac{d}{dv}\left(\frac{v^3 \nu_{e\,ad}}{\nu_{ea}}\right)\right\rangle$$

where ν_{ea} is the frequency of elastic collisions between electrons and the atoms of the bulk gas, and the angle brackets denote the averaging over the spherically symmetric velocity distribution for electrons.

According to eq. (1.22), the momentum transferred from an ion to the admixture atoms is $\frac{M}{2} w \nu_{\mathrm{i\,ad}}$. We assume that the ions are produced only from the admixture atoms and that the ion drift velocity w is large compared with their thermal velocity. The frequency $\nu_{\mathrm{i\,ad}}$ of collisions between ions and the admixture atoms corresponds to the relative collision velocity equal to the ion drift velocity. Using eq. (4.16) for the drift velocity of heavy ions, we obtain

$$w = \frac{3\sqrt{\pi} eE}{8\nu_{\mathrm{ia}}}$$

where $\nu_{\mathrm{ia}} = N_{\mathrm{a}} \sqrt{\frac{2T}{m}}\, \bar{\sigma}$ is the frequency of collisions between the ions and the atoms of the bulk gas. Hence, the force with which an ion acts on the admixture atoms is

$$\frac{3\sqrt{\pi}}{16} eE \frac{\nu_{\mathrm{i\,ad}}}{\nu_{\mathrm{ia}}}$$

Summation of the above results yields the force acting on an admixture atom in the positive discharge column:

$$f = \frac{N_{\mathrm{e}}}{N_{\mathrm{ad}}} eE \left[\frac{1}{3} \left\langle \frac{1}{v^2} \frac{d}{dv} \left(\frac{v^3 \nu_{\mathrm{e\,ad}}}{\nu_{\mathrm{ea}}} \right) \right\rangle - \frac{3\sqrt{\pi}}{16} \frac{\nu_{\mathrm{i\,ad}}}{\nu_{\mathrm{ia}}} \right].$$

The direction from the cathode to the anode is taken here as the positive direction of the coordinate system in which the expression for the force is written. We see that the force is independent of the number density of the admixture atoms and is proportional to the degree of plasma ionization $N_{\mathrm{e}}/N_{\mathrm{a}}$. In this case it is more likely that ions act with a greater force on the admixture atoms, so that the maximum density of the admixture atoms is at the cathode. According to the similarity relation (6.10), for the constant composition of the gas mixture ($N_{\mathrm{ad}}/N_{\mathrm{a}} = \mathrm{const}$) the characteristic distance along the tube axis at which the number density of the admixture atoms varies noticeably is proportional to the discharge tube radius r_0.

§ 6.2. THE POSITIVE COLUMN OF A LOW-PRESSURE DISCHARGE

By definition of the low-pressure discharge, the mean free path of ions in it is large compared with the characteristic dimensions of the discharge. This relationship affects the characteristics of such discharges which are discussed below.

Problem 6.14. Find the density of the current of charged particles and the potential variation in the positive column of the low-pressure discharge between two infinite parallel plates. Assume that the plasma of the positive column is quasineutral, the electrons are at thermodynamic equilibrium, and the electron temperature is much higher than the temperature of atoms.

Since the transverse dimensions of this system are large, the plasma parameters depend on the longitudinal coordinate, so that we have to solve the one-dimensional problem. As electrons are at thermodynamic equilibrium, their number density at a given point is given by the Boltzmann distribution, that is,

$$N_e = N_0 \exp(eU/T_e)$$

where T_e is the electron temperature, and U is the electric field potential at the given point. The symmetry of the system of two plane electrodes makes the electric field potential $U(x)$ symmetric with respect to the point $x = 0$. Then, we have $U(x) > 0$, since otherwise the slow ions produced in the positive discharge column would be unable to escape from the discharge region, while there would be no such limitation for electrons and quasineutrality of plasma would be violated. The electric field potential in the positive column of gas discharge is shown in Fig. 6.1.

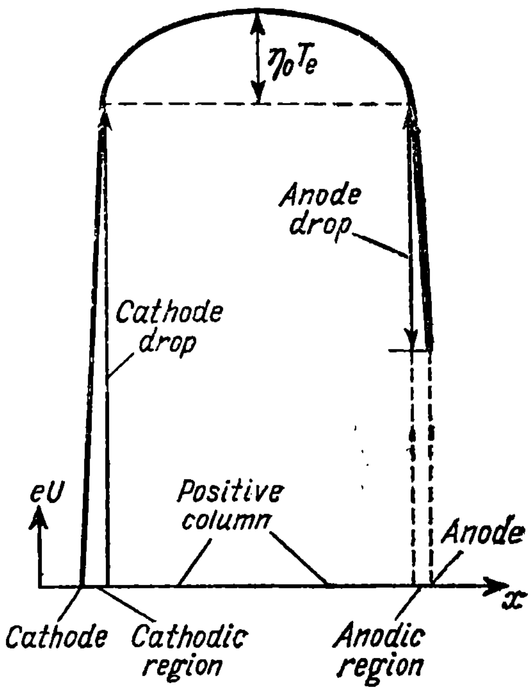

Fig. 6.1. Potential distribution in the positive column of the low-pressure electric arc.

Since the electron temperature is much higher than the gas temperature, ionization produces slow ions. The ion velocity at the point x is determined by the energy taken from the external electric field, and if the ion is produced at the point ξ, the energy is $\sqrt{2e[U(\xi) - U(x)]/M}$, where M is the ion mass.

Hence, the ion number density is

$$N_i = \int_0^x \frac{\varphi(\xi)\, d\xi}{\sqrt{2e[U(\xi) - U(x)]/M}}$$

where $\varphi(\xi)$ is the number of ions produced in the unit volume at the point ξ per unit time and the point $x = 0$ corresponds to the minimum potential of the self-consistent plasma field, so that the ions produced in the range between x and 0 and escape to the appropriate electrode pass through the point x.

The condition of plasma quasineutrality yields the equation for the self-consistent field potential

$$N_0 \exp\left(\frac{eU(x)}{T_e}\right) = \int_0^x \frac{\varphi(\xi)\,d\xi}{\sqrt{2e[U(\xi)-U(x)]/M}} \tag{6.12}$$

where N_0 is the electron number density at the point $x = 0$, and the self-consistent field potential is defined so that $U(0) = 0$. Solving eq. (6.12), we can find the distribution of the self-consistent field potential and the current density of charged particles.

Rewrite eq. (6.12) in the dimensionless form using the new variables

$$\eta(x) = -\frac{eU(x)}{T_e}, \quad j = N_0\sqrt{\frac{2T_e}{M}}$$

We obtain (for $\eta > 0$)

$$j \exp(-\eta) = \int_0^x \frac{\varphi(\xi)\,d\xi}{\sqrt{\eta(x)-\eta(\xi)}}$$

Multiply this equation by $\dfrac{d\eta}{dx}\dfrac{1}{\sqrt{\eta(y)-\eta(x)}}$ and integrate the result over dx between ξ and y. Since we have

$$\int_\xi^y \frac{\dfrac{d\eta(x)}{dx}\,dx}{\sqrt{[\eta(y)-\eta(x)][\eta(x)-\eta(\xi)]}} = \int_{\eta(\xi)}^{\eta(y)} \frac{d\eta}{\sqrt{[\eta(y)-[\eta]\,[\eta-\eta(\xi)]}} = \pi$$

we obtain for the right-hand side

$$\int_0^{\eta(y)} \frac{d\eta(x)}{\sqrt{\eta(y)-\eta(x)}} \int_0^{\eta(x)} \frac{\varphi(\xi)\,d\eta(\xi)}{\sqrt{\eta(x)-\eta(\xi)}\,\dfrac{d\eta(\varepsilon)}{d\varepsilon}}$$

$$= \int_0^{\eta(y)} \frac{\varphi(\xi)\,d\eta(\xi)}{\dfrac{d\eta}{d\xi}} \int_{\eta(\xi)}^{\eta(y)} \frac{d\eta(x)}{\sqrt{[\eta(y)-\eta(x)]\,[\eta(x)-\eta(\xi)]}} = \pi \int_0^y \varphi(\xi)\,d\xi$$

Finally, we obtain

$$I \equiv \int_0^y \varphi(\xi)\,d\xi = \frac{j}{\pi}\int_0^{\eta(y)} \exp(-\eta)\frac{d\eta}{\sqrt{\eta(y)-\eta}} \tag{6.13}$$

Equation (6.13) expresses the flux of charged particles at a given point in terms of the self-consistent field potential.

The flux I of charged particles towards the electrodes steadily increases with increasing distance from the point $x = 0$ towards one of the electrodes. Therefore, the condition $dI/d\eta = 0$ corresponds to the condition

$$\frac{d\eta}{dx} = \infty, \quad \frac{dI}{d\eta} = \frac{dI}{dx}\frac{dx}{d\eta}$$

that is, it determines the point where plasma quasineutrality is violated. Hence, the condition $dI/d\eta = 0$ is satisfied at the electrode.

Let us find $\eta = \eta_0$ at the point where $dI/d\eta = 0$. This quantity describes the variation of the self-consistent field potential from the centre of the positive discharge column to the electrode. Equation (6.13) shows that η_0 is the solution of the equation

$$\sqrt{\eta_0}\int_0^{\eta_0} \frac{\exp(-\eta)\,d\eta}{\sqrt{\eta_0-\eta}} = 1$$

and we obtain $\eta_0 = 0.855$. The current density at this point is

$$I_0 = \frac{j}{\pi\sqrt{\eta_0}} = 0.344j = 0.344N_0\sqrt{\frac{2T_e}{M}} \tag{6.14}$$

Problem 6.15. Find a criterion for the applicability of the results of Problem 6.14 related to plasma quasineutrality and thermodynamic equilibrium between electrons.

As follows from the Poisson equation, the plasma of the positive discharge column is quasineutral if

$$|N_i - N_e| = \left|\frac{1}{4\pi e}\frac{d^2U}{dx^2}\right| \ll N_0$$

or

$$\frac{d^2\eta}{dx^2} \ll \frac{1}{r_D^2} \tag{6.15}$$

where r_D is the Debye-Hückel radius (2.7). The value of η in the positive column region varies by about unity. Therefore, the results are valid if the size l of the positive column is much larger than the Debye-Hückel radius r_D, which is practically always the case.

Electrons are at thermodynamic equilibrium if the frequency ν_{esc} of the escape of electrons to the electrodes is much lower than the frequency of collisions between electrons. Owing to quasineutrality of plasma, the escape velocity for electrons is equal to the velocity $v_i \sim \sqrt{T_e/M}$ with which the ions escape to the electrodes. Therefore, the frequency of the escape of electrons to the electrodes is of the order of $\frac{1}{L}\sqrt{\frac{T_e}{M}}$, where L is the distance between the electrodes.

The frequency of collisions between electrodes $\nu_{ee} \sim \sqrt{\frac{T_e}{m}}\, N_e \frac{e^4}{T_e^2} \ln \Lambda$, where $\ln \Lambda$ is the Coulomb logarithm [see eq. (2.36 a)]. Hence, the results are valid for not too low electron number densities:

$$N_e \gg \sqrt{\frac{m}{M}}\,\frac{T_e^2}{e^4 L \ln \Lambda} \tag{6.16}$$

Problem 6.16. Find the anode voltage drop for the low-pressure discharge between two parallel electrodes discussed in Problem 6.14.

The currents of ions produced in the positive discharge column and flowing to the anode and cathode are identical. The total ion current density is

$$j_i = 2I_0 = 0.69\sqrt{\frac{2T_e}{M}}\, N_0$$

It is equal to the electron current density, so that we have

$$j_e = \int_{v_0}^{\infty} f(v_x)\, v_x\, dv_x = 0.69\sqrt{\frac{2T_e}{M}}\, N_0$$

where $f(v_x)$ is the velocity distribution for electrons, v_0 is determined by the anode voltage drop U_a, and $mv_0^2/2 = eU_a$. Assume that the velocity distribution for electrons has the Maxwellian form:

$$f(v_x) = N_0 \left(\frac{m}{2\pi T_e}\right)^{1/2} \exp\left(-\frac{mv_x^2}{2T_e}\right)$$

Now we obtain

$$j_e = N_0 \sqrt{\frac{T_e}{2\pi m}} \exp\left(-\frac{eU_a}{T_e}\right) = 0.69\sqrt{\frac{2T_e}{M}}\, N_0$$

and hence

$$eU_a = T_e \ln (0.41 \sqrt{M/m})$$

This equation shows that the anode voltage drop U_a is greater by a factor of three-five than the electron temperature, depending on the ion mass.

Problem 6.17. Find the variation of the density of the current of charged particles at the walls owing to the ion-atom resonant charge transfer in the conditions of Problem 6.14. Assume that the mean free path of ions with respect to charge transfer is large compared with the distance between the electrodes and the number density of atoms is uniform over the cross section of the system.

We shall use the same approach as in Problem 6.14. First, derive an equation for plasma quasineutrality similar to eq. (6.12), taking into account that slow ions are produced not only by ionization of atoms but also with ion-atom resonant charge transfer. This process results in stopping of ions since after the charge transfer the ion has the velocity of the atom which loses its electron, and the characteristic velocity of atoms is much lower than the characteristic velocity of ions. Moreover, the probability that an ion is not involved in a charge transfer after passing the distance x is $\exp(-x/\lambda)$, where $\lambda = (N_a\sigma_{res})^{-1}$ is the mean free path of ions with respect to charge transfer, N_a is the number density of atoms, and σ_{res} is the cross section of the resonant charge transfer. Now we can rewrite eq. (6.12) in the form

$$N_0 \exp\left[\frac{eU(x)}{T_e}\right] = \int_0^x \frac{[\varphi_{ion}(\xi) + \varphi_{res}(\xi)] \exp\left(\frac{\xi - x}{\lambda}\right)}{\sqrt{2e[U(\xi) - U(x)]/M}} d\xi$$

where $\varphi_{ion}(\xi)$ and $\varphi_{res}(\xi)$ are the numbers of slow ions produced at the point ξ per unit volume per unit time owing to ionization and resonant charge transfer, respectively. Using the same notation as in Problem 6.14, we write

$$\eta(\xi) = -\frac{eU(\xi)}{T_e}, \quad j = N_0\sqrt{\frac{2T_e}{M}}$$

and obtain

$$j \exp(-\eta) = \int_0^x \frac{[\varphi_{ion}(\xi) + \varphi_{res}(\xi)] \exp\left(\frac{\xi - x}{\lambda}\right) d\xi}{\sqrt{\eta(x) - \eta(\xi)}}$$

The parameters $\varphi_{\text{ion}}(\xi)$ and $\varphi_{\text{res}}(\xi)$ are linked. The variation of the number of ions produced by charge transfer at the distance dx is proportional to the number of the ions produced by ionization:

$$d\varphi_{\text{res}} = \varphi_{\text{ion}}(x)\frac{dx}{\lambda}$$

This yields

$$\varphi_{\text{res}}(\xi) = \frac{1}{\lambda}\int_0^{\xi} \varphi_{\text{ion}}(\xi')\,d\xi'$$

Using this expression, we can rewrite the equation in the form

$$j\exp(-\eta) = \exp(-x/\lambda)\int_0^x \frac{\dfrac{d}{d\xi}\left[\exp\left(\dfrac{\xi}{\lambda}\right)\displaystyle\int_0^{\xi}\varphi_{\text{ion}}(\xi')\,d\xi'\right]}{\sqrt{\eta(x)-\eta(\xi)}}\,d\xi$$

Using the same procedure for solving this equation (*Abel's equation*) as in Problem 6.14, we obtain

$$\frac{j}{\pi}\int_0^{\eta}\frac{d\eta'}{\sqrt{\eta-\eta'}}\exp\left[-\eta'+\frac{x(\eta')}{\lambda}\right] = \exp(x/\lambda)\int_0^x \varphi_{\text{ion}}(\xi)\,d\xi$$

Hence, the current density at the point x is

$$I = \int_0^x \varphi_{\text{ion}}(\xi)\,d\xi = \frac{j}{\pi}\int_0^{\eta}\frac{d\eta'}{\sqrt{\eta-\eta'}}\exp\left[-\eta'+\frac{x(\eta')}{\lambda}-\frac{x(\eta)}{\lambda}\right]$$

Since the current density is maximal at the electrode, the condition $dI/d\eta = 0$ is satisfied at the electrode. Hence, using the expression for the derivative

$$\frac{d}{d\eta}\int_0^{\eta}\frac{f(\eta')\,d\eta'}{\sqrt{\eta-\eta'}} = \int_0^{\eta}\frac{\dfrac{df}{d\eta'}\,d\eta'}{\sqrt{\eta-\eta'}} + \frac{f(0)}{\sqrt{\eta}}$$

we obtain the following equation for $\eta_0 = \eta(l)$:

$$\frac{1}{\sqrt{\eta_0}} = \int_0^{\eta_0}\frac{\left(1-\dfrac{1}{\lambda}\dfrac{dx'}{d\eta'}\right)\exp\left[-\eta'+\dfrac{x(\eta')}{\lambda}\right]}{\sqrt{\eta_0-\eta'}}\,d\eta'$$

The resulting current density at the electrodes is

$$I_{\max} = \int_0^l \varphi_{\text{ion}}(x)\,dx = \frac{j}{\pi}\,\frac{\exp\left(-\frac{l}{\lambda}\right)}{\sqrt{\eta_0}} + \frac{j}{\pi}\int_0^l \frac{dx}{\lambda}\,\frac{\exp\left(-\eta-\frac{l}{\lambda}+\frac{x}{\lambda}\right)}{\sqrt{\eta_0-\eta}}$$

In the limit $\lambda \to \infty$ all the above results coincide with those derived in Problem 6.14.

Let us find the current density at the electrodes to the first-order term in the small parameter l/λ. We obtain

$$I_{\max} = \frac{j}{\pi\sqrt{\eta_0}}\left[1-\frac{l}{\lambda}\left(1-\sqrt{\eta_0}\int_0^l \frac{dx}{l}\,\frac{\exp(-\eta)}{\sqrt{\eta_0-\eta}}\right)\right]$$

where $j/\pi\sqrt{\eta_0}$ is the current density in the absence of a resonant charge transfer, and $\eta_0 = 0.855$. To calculate the last integral, we have to find $\eta(x)$. Let us take a simple approximation for this parameter, giving a correct description of its behaviour. Since for $x = 0$ we have $\eta = 0$ and $d\eta/dx = 0$ and for $x = l$ we have $\eta = \eta_0$ and $d\eta/dx = \infty$, we can write

$$\left(\frac{\eta-\eta_0}{\eta_0}\right)^2 + \left(\frac{x}{l}\right)^2 = 1$$

In this case we obtain for the dimensionless potential at the electrodes $\eta_0 = 0.855$ and for the current density at the electrodes

$$I_{\max} = \frac{j}{\pi\sqrt{\eta_0}}\left[1-\frac{l}{\lambda}\left(1-\eta_0\int_0^1 \frac{\exp(-t\eta_0)\sqrt{1-t}\,dt}{\sqrt{t(2-t)}}\right)\right]$$

Finally, we obtain

$$I_{\max} = 0.344j(1-l/\lambda)$$

We see that the effect of the resonant charge transfer is to decrease the current density at the electrodes.

Problem 6.18. **Find the flux of charged particles at the walls of a cylindrical discharge tube and the variation of the potential from the axis to the walls of the tube in the positive column of a low-pressure gas discharge.**

Assume that the frequency of electron-electron collisions is much higher than the frequency of escape of charged particles to the tube walls. Therefore, electrons are at thermodynamic equilibrium and their number density at the distance ρ from the tube axis is given by the Boltzmann distribution

$$N_e(\rho) = N_0 \exp \left\{ \frac{-e[U_0 - U(\rho)]}{T_e} \right\}$$

where U_0 is the electric field potential at the tube axis, T_e is the electron temperature, and N_0 is the electron number density at the tube axis.

Assuming that the electron temperature is much higher than the temperature of atoms, we can write the following equation for the ion number density:

$$N_i(\rho) 2\pi\rho = \int_0^\rho \frac{\varphi(r)\, 2\pi r\, dr}{\sqrt{2e[U(r) - U(\rho)]/M}}$$

Here $\varphi(r)$ is the frequency of production of ions per unit volume. Since the positive column plasma is quasineutral, we can equate the number densities of ions and electrons and obtain

$$2\pi\rho N_0 \exp \left\{ \frac{-e[U_0 - U(\rho)]}{T_e} \right\} = \int_0^\rho \frac{\varphi(r)\, 2\pi r\, dr}{\sqrt{2e[U(r) - U(\rho)]/M}}$$

Using the new variables $\eta(\rho) = -e[U(\rho) - U_0]/T_e$ and $\omega = N_0\sqrt{2T_e/M}$, we can rewrite the equation for the self-consistent field potential in the form

$$\omega\rho \exp[-\eta(\rho)] = \int_0^\rho \frac{\varphi(r)\, r\, dr}{\sqrt{\eta(\rho) - \eta(r)}} \qquad (6.17)$$

We have to find the electron current per unit length at the tube wall

$$I = \int_0^{r_0} \varphi(r)\, 2\pi r\, dr$$

where r_0 is the discharge tube radius. The current density at the tube wall is

$$j = \frac{I}{2\pi r_0} = \int_0^{r_0} \varphi(r) r \frac{dr}{r_0} \qquad (6.18)$$

Multiply both sides of eq. (6.18) by $[\eta(R) - \eta(\rho)]^{-1/2}\, d\eta/d\rho$ and integrate the result over $d\rho$ between 0 and R. Since we have

$$\int_{\eta_r}^{\eta_R} \frac{d\eta}{\sqrt{(\eta_R - \eta)(\eta - \eta_r)}} = \pi$$

we obtain

$$\omega\int_0^{\eta_R}\frac{\rho\,d\eta}{\sqrt{\eta_R-\eta}}\exp(-\eta)=\pi\int_0^R\varphi(r)r\,dr$$

This equation can be used to find the current density at the walls given by eq. (6.18), which is

$$j=\int_0^{r_0}\varphi(r)\frac{r\,dr}{r_0}=\frac{\omega}{\pi_{r_0}}\int_0^{\eta_0}\frac{\rho\,d\eta\exp(-\eta)}{\sqrt{\eta_0-\eta}}$$

where $\eta_0=\eta(r_0)$.

At the discharge tube walls the self-consistent field potential varies by an amount of the order of temperature. This variation is needed to equate the currents of ions and electrons; it occurs at the distances of the order of the Debye-Hückel radius, where charges are separated and the plasma ceases to be quasineutral. Since $\frac{dj}{dr}=\frac{dj}{d\eta}\frac{d\eta}{dr}$ and $\frac{dj}{dr}$ is finite, while the derivative $\frac{d\eta}{dr}=\infty$ at the tube walls under the conditions of this problem, we obtain $\frac{dj}{d\eta}=0$ at the walls. This condition for η_0 at the tube walls yields

$$\frac{d}{d\eta}\int_0^{\eta}\frac{\rho\,d\xi\exp(-\xi)}{\sqrt{(\eta-\xi)}}\Bigg|_{\eta=\eta_0}=0 \tag{6.19}$$

and for $\eta=\eta_0$ we have $\rho=r_0$ and $d\rho/d\eta=0$. Moreover, in the vicinity of the tube axis the electric field $E\sim\rho$, that is, for $\rho\to0$ we have $\eta\sim\rho^2$. At the same time, η is a monotonic function of ρ, that is, $d\eta/d\rho>0$. The simplest function satisfying these conditions is

$$\frac{\rho}{r_0}=\frac{\sqrt{\eta(2\eta_0-\eta)}}{\eta_0},\quad\frac{\eta}{\eta_0}=1-\sqrt{1-\left(\frac{\rho}{r_0}\right)^2} \tag{6.20}$$

Assuming that this function is a close approximation, we can find the following equation for η_0 from eq. (6.19) for any $\varphi(r)$:

$$\int_0^{\eta_0}\left[\sqrt{\frac{\eta(2\eta_0-\eta)}{\eta_0-\eta}}-\sqrt{\frac{\eta_0-\eta}{\eta(2\eta_0-\eta)}}\right]\exp(-\eta)\,d\eta=0$$

This equation yields $\eta_0 = 1.145$ and

$$j = \frac{\omega}{\pi}\sqrt{\eta_0}\int_0^1 \sqrt{\frac{t(2-t)}{1-t}}\exp(-t\eta_0)\,dt = 0.854\frac{\omega}{\pi} = 0.272N_0\sqrt{\frac{2T_e}{M}} \qquad (6.21)$$

Problem 6.19. Find the velocity distribution for ions along the tube axis in a cylindrical discharge tube. Assume that the electric field E at the tube axis is constant and uniform and the frequency of production of charged particles at each point is proportional to the electron number density at this point.

The ion velocity parallel to the tube axis is eEt/M, where t is the time after the production of an ion. The time needed for an ion produced at the point r to reach the point r_1 is

$$t = \int_r^{r_1}\frac{d\rho}{v_\rho} = \left(\frac{M}{2T_e}\right)^{1/2}\int_r^{r_1}\frac{d\rho}{\sqrt{\eta(\rho)-\eta(r)}}$$

$$= \left(\frac{M}{2T}\right)^{1/2} r_0\int_\eta^{\eta_1}\frac{\eta_0-\eta'}{\eta_0}\,\frac{d\eta}{\sqrt{\eta'(\eta-\eta')\,(2\eta_0-\eta')}}\,.$$

The time needed for an ion produced at the point r to reach the tube wall is

$$t_{\max} = \left(\frac{M}{2T_e}\right)^{1/2} r_0\int_\eta^{\eta_0}\sqrt{\frac{\eta_0-\eta}{\eta(2\eta_0-\eta)}}\,\frac{d\eta}{\eta_0}$$

For instance,

$$t_{\max}(0) = \left(\frac{M}{2T_e}\right)^{1/2} r_0\frac{1}{\sqrt{\eta_0}}\int_0^1\sqrt{\frac{1-x}{x(2-x)}}\,dx = 1.12\left(\frac{M}{2T_e}\right)^{1/2} r_0$$

We see that the longitudinal velocities of the ions produced at the point ρ are in the range $0 \leqslant v_z \leqslant \frac{eE}{M}t_{\max}$. The number of ions produced at the point ρ and having the velocities in the range from v_z to $v_z + dv_z$ is proportional to dv_z and independent of v_z. This is explained by the continuous mechanism of ion production. Hence, the number of ions with the velocity v_z is proportional to

$$\int_0^r \varphi(r')r'\,dr' = C\{(\eta_0 - 1)[1 - \exp(-\eta)] + \eta\exp(-\eta)\}$$

where v_z and η are related by

$$v_z = \frac{eE}{m} t_{\max}(r) = v_0 \int_0^{\eta_0} \sqrt{\frac{\eta_0 - \eta'}{\eta'(2\eta_0 - \eta')}} \frac{d\eta'}{\sqrt{\eta_0}}, \quad v_0 = \frac{eEr_0}{\sqrt{2T_e M\eta_0}}$$

Hence, we have

$$dv_z = v_0 \sqrt{\frac{\eta_0 - \eta}{\eta(2\eta_0 - \eta)\eta_0}}\, d\eta$$

and

$$f(v_z)\,dv_z = C\{(\eta_0 - 1)[1 - \exp(-\eta)] + \eta\exp(-\eta)\} \sqrt{\frac{\eta_0 - \eta}{\eta(2\eta_0 - \eta)\eta_0}}\, d\eta$$

where the constant C is found from the normalization condition for the distribution function. The mean ion velocity parallel to the discharge tube axis is

$$\bar{v}_z = \frac{\int_0^{\eta_0} \{(\eta_0-1)[1-\exp(-\eta)]+\eta\exp(-\eta)\} \sqrt{\frac{\eta_0 - \eta}{(2\eta_0-\eta)\eta\eta_0}}\, d\eta \int_\eta^{\eta_0} \frac{d\eta'}{\sqrt{\eta_0}} \sqrt{\frac{\eta_0 - \eta'}{\eta'(2\eta_0-\eta')}}}{\int_0^{\eta_0} \{(\eta_0 - 1)[1 - \exp(-\eta)] + \eta\exp(-\eta)\} \sqrt{\frac{\eta_0 - \eta}{(2\eta_0 - \eta)\eta\eta_0}}\, d\eta}$$

$$= \frac{0.08707}{0.2539} v_0 = 0.343 v_0$$

Problem 6.20. Find the distribution of ions over the transverse velocities in the cylindrical discharge tube. Assume that the frequency of production of ions at each point is proportional to the number density of free electrons.

The number density of ions with the energies in the range from ε to $\varepsilon + d\varepsilon$ at the point ρ is

$$f(\rho, \varepsilon)\, d\varepsilon = \frac{\varphi(r')r'\, dr'}{\sqrt{\eta(\rho) - \eta(r')}}$$

where $\varepsilon/T_e = \eta(\rho) - \eta(r')$. This relationship determines the point r'; the ions coming from this point have the energy ε at the point ρ and, hence, the radial velocity $v_\rho = \sqrt{2\varepsilon/M}$. Substituting into the equation

$$\varphi(r') = C \exp(-\eta'), \quad \frac{\rho}{r_0} = \sqrt{\frac{\eta(2\eta_0 - \eta)}{\eta_0}}$$

we obtain

$$f(\rho, \varepsilon)\, d\varepsilon = C\, \frac{\exp[\varepsilon/T_e - \eta]\,(\eta_0 + \varepsilon/T_e - \eta)}{\sqrt{\varepsilon/T_e}}\, d\varepsilon$$

The constant C can be found from the normalization condition for the distribution function f at the tube wall:

$$\int_0^{\eta_0 T_e} f(r_0, \varepsilon)\, d\varepsilon = N_0 \exp(-\eta_0)$$

Problem 6.21. Find the voltage drop at the wall of the cylindrical discharge tube for the low-pressure arc.

The voltage drop occurs at a small region at the wall with the size of the order of the Debye-Hückel radius. In this region the positive and negative charges are separated and the resulting potential difference balances the ion and electron current at the tube wall, conserving quasineutrality of plasma in the tube. The electron current density at the wall is

$$j_e = N_0 \exp(-\eta_0) \int_{v_0}^{\infty} \exp\left[-\frac{mv_x^2}{2T_e}\right] v_x\, dv_x \sqrt{\frac{m}{2\pi T_e}}$$

$$= N_0 \exp\left[-\eta_0 - \frac{e\Delta U}{T_e}\right]\sqrt{\frac{T_e}{2\pi m}}$$

where $\Delta U = mv_0^2/2e$ is the voltage drop at the wall, and the x-axis is perpendicular to the wall. Equating the electron current density and the ion current density which is $j_i = 0.272 N_0 \sqrt{2T_e/M}$, we obtain the following equation for the voltage drop:

$$\exp\left[-\eta_0 - \frac{e\Delta U}{T_e}\right] = 0.54\sqrt{\frac{\pi m}{M}}, \quad \frac{e\Delta U}{T_e} = \ln\left(\frac{1}{3}\sqrt{\frac{M}{m}}\right) = c \qquad (6.22)$$

For instance, for helium $c = 3.35$, for argon $c = 4.5$, and for xenon $c = 5.1$.

Problem 6.22. Find the temperature and current of electrons in a cylindrical discharge tube if the longitudinal electric field is uniform in the cross section of the tube.

On the average, a pair of charged particles is produced by each electron during its lifetime in the discharge tube. Since the cross section for inelastic collisions between electrons and atoms is considerably smaller than the cross section for elastic collisions, the electron undergoes numerous elastic collisions with atoms during its lifetime. Therefore, the momentum acquired by the electron from the electric field is transferred to atoms. The balance equation for the variation of the electron momentum per unit time can be written as

$$\frac{d\mathbf{P}}{dt} = e\mathbf{E} - m\langle \mathbf{v}\nu_{ea}\rangle = 0$$

where the angle brackets denote the averaging over the Maxwellian distribution function for electrons, $\nu_{ea} = N_a v_e \sigma^*_{ea}$ is the frequency of the electron-atom collisions, N_a is the number density of atoms, and σ^*_{ea} is the diffusion cross section of electron scattering by an atom. Assuming that the electron drift velocity w is much lower than their thermal velocity, we obtain $\langle \mathbf{v}\nu_{ea}\rangle = \mathbf{w}\langle \nu_{ea}\rangle$. Hence, we have

$$w = \frac{eE}{m\langle \nu_{ea}\rangle} = \frac{eE\lambda_{ea}}{\sqrt{2T_e m}}$$

where $\lambda_{ea} = \sqrt{2T_e}/(\langle \nu_{ea}\rangle \sqrt{m})$ is the mean free path of electrons.

Let us write the balance equation for the variation of the electron energy per unit time. The energy lost by each electron is

$$\tilde{\varepsilon} = \bar{\eta}T_e + \bar{\varepsilon} + I + \bar{\eta}T_e + e\Delta U$$

where $\bar{\eta}T_e$ is the mean energy spent by an electron to approach the wall, I is the ionization potential, that is, the energy required to produce a pair of charged particles, and $\bar{\varepsilon}$ is the mean energy transported by the electron to the wall. The last two terms describe the kinetic energy of ions acquired owing to the interaction with the electric field: $\bar{\eta}T_e$ is the mean kinetic energy acquired by the ion when approaching the wall, and ΔU is the potential drop at the wall. We have

$$\bar{\eta} = \frac{\int_0^{r_0} \exp(-\eta)\,\eta\rho\,d\rho}{\int_0^{r_0} \exp(-\eta)\,\rho\,d\rho} = \frac{\int_0^{\eta_0} \exp(-\eta)\,\eta(\eta_0-\eta)\,d\eta}{\int_0^{\eta_0} \exp(-\eta)\,(\eta_0-\eta)\,d\eta}$$

$$= \frac{\eta_0 - 1 + \exp(-\eta_0)\left(1-\frac{\eta_0^2}{2}\right)}{\eta_0 - 1 + \exp(-\eta_0)} = 0.575$$

$$\bar{\varepsilon} = T_e + \frac{\int_{v_0}^{\infty} \exp\left[-\frac{mv_x^2}{2T_e}\right]\frac{mv_x^2}{2}\,dx}{\int_{v_0}^{\infty} \exp\left[-\frac{mv_x^2}{2T_e}\right]dv_x} \approx T_e + e\,\Delta U + T_e = e\,\Delta U + 2T_e$$

$$\tilde{\varepsilon} = 2\bar{\eta}T_e + e\Delta U + \bar{\varepsilon} + I \approx 3.1T_e + 2e\Delta U + I$$

Now the balance equation for the variation of the total electron energy per unit time is

$$eEw\int_0^{r_0} N_e\,2\pi\rho\,d\rho\,L = 2\pi r_0 L j\tilde{\varepsilon}$$

where L is the tube length. Hence, we obtain

$$eEw = \frac{0.76\tilde{\varepsilon}}{r_0}\sqrt{\frac{2T_e}{M}}$$

Using both balance equations, we find

$$w = \left[\frac{0.76\tilde{\varepsilon}}{\sqrt{Mm}}\frac{\lambda_{ea}}{r_0}\right]^{1/2}, \quad \frac{eE\sqrt{r_0\lambda_{ea}}}{T_e} = \left(\frac{m}{M}\right)^{1/4}\left(\frac{1.52\tilde{\varepsilon}}{T_e}\right)^{1/2}$$

Compare the electron drift velocity with their thermal velocity. Since $\tilde{\varepsilon} \sim T_e$, we have

$$w\sqrt{\frac{m}{T_e}} \sim \sqrt{\frac{\lambda_{ea}}{r_0}}\left(\frac{m}{M}\right)^{1/4}$$

Since $r_0\sqrt{M/m}$ is the distance travelled by the electron during its lifetime, we have $w \ll \sqrt{T_e/m}$.

The mean longitudinal velocity of ions is $\bar{v}_i = 0.34eEr_0/\sqrt{2TM}\eta_0$. Compare it with the electron drift velocity:

$$\frac{\bar{v}_i}{w} \sim \frac{eEr_0}{T_e}\left(\frac{m}{M}\right)^{1/4}\left(\frac{r_0}{\lambda_{ea}}\right)^{1/2} \sim \left(\frac{m}{M}\right)^{1/2}\frac{r_0}{\lambda_{ea}} \ll 1$$

(each of the factors is smaller than unity). The total longitudinal current in the tube is

$$I = \int N_e\, 2\pi\rho\, d\rho\, w = wN_0 \cdot 0.71\pi r_0^2$$

Problem 6.23. Find the relationship between the number density of atoms and the electron temperature in the cylindrical discharge tube.

The balance equation for the electron number density yields

$$\frac{d}{dt}\int_0^{r_0} N_e \cdot 2\pi\rho\, d\rho = 0 = k_{\text{ion}}N_a\int_0^{r_0} 2\pi\rho\, d\rho\, N_e - 2\pi r_0 \cdot 0.27N_0\sqrt{\frac{2T_e}{M}}$$

$$k_{\text{ion}}N_a = \frac{0.76}{r_0}\sqrt{\frac{2T_e}{M}}$$

Hence, the number density of atoms is

$$N_a = \frac{0.76}{r_0 k_{\text{ion}}(T_e)}\sqrt{\frac{2T_e}{M}} \tag{6.23}$$

Equation (6.23) determines the electron temperature needed to maintain discharge in the case of a weakly ionized plasma. We see that the electron temperature remains constant when the gas density and the tube radius are varied if $N_a r_0 = \text{const}$. According to the results of Problem 6.22, under these conditions we have $eEr_0 \sim \sqrt{1/N_a r_0} \sim \text{const}$, that is, the electric field varies as $1/r_0$ (the frequency of the electron-atom collisions is much higher than the frequency of electron escape to the walls).

The electron number density in the discharge tube is determined by the external factors, namely, by the parameters of the external source generating the discharge. Since the electron temperature is fully determined by the gas density and the tube radius, these parameters also determine the electric field. Thus, for the given tube radius and the gas density a definite voltage should be applied to the discharge tube. Under these conditions the current in the discharge tube depends on the parameters of the external source, and the conductivity of the discharge gap is proportional to $N_e r_0^2$.

Problem 6.24. Find the longitudinal gradient of the gas density in a sealed cylindrical discharge tube.

The gradient of the gas density in the discharge tube is caused by the current of charged particles. Collisions between charged particles and gas atoms give rise to a directed flow of gas atoms. Let the mean velocity of gas atoms along the tube axis be $\mathbf{w}_a$. Since the directed flow is compensated by the diffusion flow, the equation for the number density of gas atoms is

$$\mathscr{D}\nabla N_a - \mathbf{w}_a N_a = 0$$

where N_a is the number density of atoms, and $\mathscr{D}$ is their diffusion coefficient. If the pressure difference is small, we have

$$\Delta N/N = w_a L/\mathscr{D}$$

where ΔN is the difference between the gas densities at the electrodes, and L is the distance between the electrodes.

Let us find the translational velocity of atoms due to their collisions with electrons and ions. The time of escape of ions to the tube walls is shorter than the time of collision with atoms, so that the momentum acquired by ions from the field is fully transported to the walls. Since the frequency of electron-atom elastic collisions is considerably higher than the frequency of electron escape to the walls, each electron transfers to the gas atoms the momentum eE per unit time, which it acquires from the electric field. Atoms transport this momentum to the walls, and we shall assume that the mean longitudinal momentum of the atoms reflected from the walls is zero. The balance equation for the momentum of the atoms per unit length of the discharge tube can be written as

$$\frac{dP_x}{dt} = 0 = eE\int N_e(\rho)2\pi\rho\, d\rho - Mw_a \cdot 2\pi r_0 N_a \sqrt{\frac{T_a}{2\pi M}}$$

Hence, we obtain

$$w_a = \frac{0.71\, \pi r_0^2 N_e E e\sqrt{2\pi M}}{M \cdot 2\pi r_0 N_a \sqrt{T_a}} = 0.88\frac{N_0}{N_a}\frac{eEr_0}{\sqrt{T_a M}}$$

where N_0 is the electron number density at the tube axis. Using the relationship between the longitudinal electric field and the electron temperature found in Problem 6.21,

$$\frac{eEr_0}{T_e} = \left(\frac{r_0}{\lambda_{ea}}\right)^{1/2}\left(\frac{m}{M}\right)^{1/4}\left(\frac{1.5\tilde{\varepsilon}}{T_e}\right)^{1/2}$$

we obtain

$$w_a = 1.1\frac{N_0}{N_a}\frac{T_e}{\sqrt{T_a M}}\left(\frac{r_0}{\lambda_{ea}}\right)^{1/2}\left(\frac{m}{M}\right)^{1/4}\left(\frac{\tilde{\varepsilon}}{T_e}\right)^{1/2}$$

Hence, the relative density difference is

$$\frac{\Delta N_a}{N_a} \sim \frac{w_a L}{D} \sim \frac{N_0}{N_a}\left[\left(\frac{m}{M}\right)^{1/2}\frac{r_0}{\lambda_{ea}}\right]^{1/2}\frac{\tilde{\varepsilon}}{T_a}\frac{L}{\lambda_{aa}}$$

where λ_{aa} is the mean free path of atoms. The first two factors are much smaller than unity, while the last two factors are greater than unity. Therefore, the density difference can be either large or small, depending on the system parameters.

Problem 6.25. Find the difference between the gas pressures at the axis and at the wall of a discharge tube.

The difference between the gas densities at the axis and at the walls o the discharge tube is due to the fact that the atoms in the tube are ionized that is, their density decreases. As the ions are neutralized at the walls, the number density of atoms at the wall increases. If the velocity distribution of atoms is isotropic, their flux to any point in the tube and from it is $j = N_a\bar{v}_a/4$, where $\bar{v}_a = \sqrt{8T_a/\pi M}$ is the mean velocity of atoms. The balance equation for the number density N_a of atoms at the distance ρ from the tube axis is

$$\frac{dN_a}{dt} = 0 = -\frac{dj}{d\rho} - N_e N_a k_{ion}$$

Hence, we obtain

$$\frac{dN_a}{N_a d\rho} = -\frac{4N_e k_{ion}}{\bar{v}}, \quad N_a = N_a^{(0)}\exp\left[\int_{\rho}^{r_0}\frac{4N_e k_{ion}}{\bar{v}}\,d\rho\right]$$

and the relative pressure difference is given by

$$\frac{\Delta N_a}{N_a^{(0)}} = \int_0^{r_0}\frac{4N_e k_{ion}}{\bar{v}}\,d\rho = \sqrt{\frac{8\pi M}{T_a}}N_0\int_0^{r_0}\exp(-\eta)\frac{0.76}{N_a r_0}\sqrt{\frac{2T_e}{M}}\,d\rho$$

$$= 4.35\frac{N_0}{N_a}\sqrt{\frac{T_e}{T_a}} \qquad (6.24)$$

where N_0 is the electron density at the tube axis. This equation has been derived from eq. (6.20) and the results of Problem 6.23. We see that the pressure difference is small and the distribution of atoms in the tube cross section is uniform only for a weakly ionized plasma.

Problem 6.26. Compare the size of the Langmuir layer at the discharge tube wall in which the positive and negative charges are separated with the Debye-Hückel radius.

The region at the tube wall, where plasma quasineutrality is violated, contains mostly the positive ions, The layer at the wall where charges are separated is known as the *Langmuir layer*. We have to estimate the thickness l of this layer and to compare it with the Debye-Hückel radius.

The fields in the Langmuir layer decelerate electrons and accelerate ions, thus, balancing the currents of electrons and ions at the wall, so that plasma remains quasineutral. The current of electrons and ions at the wall is determined by the internal discharge region and, according to the results of Problem 6.4, is $j = 1.25\mathscr{D}_a N_0/r_0$, where N_0 is the number density of charged particles in the central discharge region, r_0 is the discharge tube radius and $\mathscr{D}_a$ is the ambipolar diffusion coefficient. According to eq. (2.11), $\mathscr{D}_a \sim \mathscr{D}_i \dfrac{T_e}{T_i} \sim \dfrac{T_e\lambda}{\sqrt{T_i M}}$, where $\mathscr{D}_i$ is the diffusion coefficient for ions in the gas, λ is the mean free path of ions in the gas, M is the ion mass, T_e and T_i are the temperatures of electrons and ions, and $T_e \gtrsim T_i$. Hence, we find

$$j \sim T_e\lambda N_0/r_0\sqrt{T_i M}$$

The Poisson equation for the field potential φ at the wall is

$$d^2\varphi/dx^2 = 4\pi e(N_i - N_e)$$

where N_e and N_i are the number densities of electrons and ions. Since $N_i > N_e$ at the wall, we can assume that the electron number density is zero. The ion current at the wall is $j = N_i v_i =$ const as no charged particles are produced at the wall (v_i is the ion velocity at the wall). Substituting this current into the Poisson equation and using the above estimate of the ion current, we obtain

$$\frac{\Delta\varphi}{l^2} \sim 4\pi e \frac{T_e\lambda N_0}{r_0\sqrt{T_i M}\, v_i} \sim \frac{\lambda}{r_0}\,\frac{T_e}{er_D^2}\,\frac{T_e}{v_i\sqrt{T_i M}} \tag{6.25}$$

where l is the thickness of the Langmuir layer, $\Delta\varphi$ is the voltage drop at the wall (according to the results of Problem 6.21, $\Delta\varphi \sim T_e/e$) and r_D is the Debye-Hückel radius given by eq. (2.11). Below, we shall consider two limiting cases depending on the relation between the mean free path of ions and the thickness of the Langmuir layer.

If $l \ll \lambda$, the ion velocity at the wall is given by $v_i \sim \sqrt{e\Delta\varphi/M} \sim \sqrt{T_e/M}$. Substitution of this expression into eq. (6.25) yields

$$\frac{l}{r_D} \sim \left(\frac{r_0}{\lambda}\right)^{1/2}\left(\frac{T_i}{T_e}\right)^{1/4}, \quad l \ll \lambda \ll r_0 \tag{6.25a}$$

In the other limiting case, when $l \gg \lambda$, the transitional velocity of ions at the walls is determined by the ion mobility in the gas, $v_i = KE$, where, according to eq. (4.10), the mobility $K \sim e\lambda/\sqrt{T_i M}$ and the electric field at the wall $E \sim \Delta\varphi/l \sim T_e/el$. Substituting these expressions into eq. (6.25), we obtain

$$l/r_D \sim (r_0/r_D)^{1/3}, \qquad \lambda \ll l \ll r_0 \tag{6.25b}$$

We see that in both cases the thickness of the Langmuir layer at the wall is considerably greater than the Debye-Hückel radius for the central discharge region.

§ 6.3. IGNITION CONDITIONS FOR LOW-CURRENT DISCHARGES

We shall discuss below the ignition conditions for the Townsend discharges with extremely low current densities. In these discharges the space charge is small and its field does not affect the motion of charged particles.

Problem 6.27. Find the condition for self-maintained discharge between two plates of a large size. The charged particles are produced both in the gas volume and at the cathode surface under the effect of ion bombardment.

We shall describe the rate of production of charged particles in the gas volume by the first Townsend coefficient α equal to the number of pairs of charged particles produced by the electron per unit length of its path. The ions which reach the cathode surface produce free electrons at the cathode surface owing to so-called potential detachment of electrons, in which the ion transfers the energy of the order of the ionization potential to the electron.

If the ionization potential of the atom is greater than twice the work function for the cathode metal, then a free state appears in the metal which is lower than the Fermi level by the energy greater than the work function. Owing to the interaction between two electrons at the Fermi surface this state can be occupied by one of the interacting electrons, and the other one escapes.

Production of electrons at the cathode surface under the effect of ions can be described by the second Townsend coefficient γ which is the ratio of the number of free ions produced at the cathode surface to the number of ions at the cathode. The coefficient γ depends on the type of ions and the cathode metal. The physics of detachment of electrons from the cathode surface shows a weak dependence of this coefficient on the ion velocity and, hence, on the electric field at the cathode surface. Table 6.1 gives the values of the second Townsend coefficient for two ion energies and supports the above suggestion.

Table 6.1

THE SECOND TOWNSEND COEFFICIENT γ FOR COLLISIONS OF IONS WITH THE DEGASSED TUNGSTEN SURFACE

Ion	γ	
	ion energy of 1 eV	ion energy of 10 eV
H^+	0.30	0.27
Ne^+	0.21	0.25
Ar^+	0.095	0.11
Kr^+	0.048	0.06
Xe^+	0.019	0.019

Let us write the condition for self-maintained gas discharge in the gap between two plates where the electric field, and hence, the coefficient α, are spatially uniform. The continuity equation for the electron number density N_e is

$$dwN_e/dz = \alpha w N_e$$

where z is the coordinate taken from the cathode, w is the electron drift velocity, wN_e is the electron flux, and αw is the frequency of production of the pairs of charged particles under the effect of one electron.

The solution of this equation is $N_e = N_e(0)\exp(\alpha z)$, and if the distance between the electrodes is L, then each electron produced at the cathode generates $[\exp(\alpha L) - 1)]$ pairs of charged particles in the discharge gap. Each of the produced ions migrates to the cathode and produces a free electron at the cathode surface with the probability γ. For the discharge to be maintained all the ions produced by one electron at the cathode should generate one new electron. Therefore, we have

$$\gamma[\exp(\alpha L) - 1] = 1$$

or

$$\alpha L = \ln\left(1 + \frac{1}{\gamma}\right) \tag{6.26}$$

Problem 6.28. Find the dependence of the first Townsend coefficient on the electric field in weak fields.

By definition, the first Townsend coefficient is $\alpha = N_a \frac{\langle v\sigma_i \rangle}{w}$, where N_a is the number density of gas atoms, v is the electron velocity, $\sigma_i(v)$ is the cross section of the electron-impact ionization of atoms, w is the electron drift velocity, and the angle brackets denote the averaging over the velocity distribution of electrons. Since the electric field E enters the kinetic equation for electrons with low number density only in the combination eE/N_a, then the most general form of the dependence of α on E is

$$\alpha = N_a\,\varphi(eE/N_a) \tag{6.27}$$

where $\varphi(x)$ is a certain function. This dependence is valid also for high electron number densities, when the electron-electron collisions play a significant role. In this case the electron velocity distribution has the Maxwellian form for the main electron velocity range, and the dependence of the electron temperature on the electric field is expressed by eE/N_a.

In weak fields the energy distribution of electrons depends on the field as $f \sim \exp(-aE/N)$, as given by eq. (2.27), so that the strongest dependence of $\varphi(x)$ on x for small x has the form

$$\varphi(x) = C_1 \exp(-C_2/x)$$

where the factors C_1 and C_2 depend on the nature of the gas in the discharge gap.

The Townsend coefficient α is, typically, expressed as a function of the ratio E/p, where p is the gas pressure, rather than E/N, and the function has the same form for identical gas temperatures. In weak fields we have

$$\alpha = Ap \exp(-Bp/E) \tag{6.28}$$

Table 6.2 gives the factors A and B in eq. (6.28) taken from [1] to Chapter 6. These values were found by comparing the experimental results with eq. (6.28).

Table 6.2

FACTORS A AND B FROM EQ. (6.28)
(A IN $\mathrm{CM}^{-1}\cdot\mathrm{TORR}^{-1}$;
B in $\mathrm{V\cdot cm^{-1}\cdot TORR^{-1}}$;
THE APPLICABILITY RANGE R
IN $\mathrm{V\cdot cm^{-1}\cdot TORR^{-1}}$)

Gas	A	B	R	Gas	A	B	R
He	3	34	20-150	H_2	5	130	150-600
Ne	4	100	100-400	N_2	12	340	100-600
Ar	14	180	100-600	CO_2	20	470	500-1000
Kr	17	240	100-1000	H_2O	13	290	150-1000
Xe	26	350	200-800	Hg	20	370	200-600

Problem 6.29. Find the potential difference between the electrodes at which discharge is ignited, using eq. (6.26).

If the distance between the plate electrodes is L and the electric field is E, then, according to eqs. (6.26) and (6.27), the potential U_{ig} of discharge ignition is given by

$$N_a L \varphi\left(\frac{U_{ig}}{N_a L}\right) = \ln\left(1 + \frac{1}{\gamma}\right)$$

This equation yields the Paschen similarity law according to which the ignition voltage remains constant when the gas density and the distance between the electrodes are simultaneously varied, if $N_a L = \text{const}$ (naturally, for the same gas and cathode material). If the gas temperature remains constant during such variation of the gas parameters, the Paschen law can be reformulated for the gas pressure p and the interelectrode distance L.

For instance, if the Townsend coefficient is given by eq. (6.28), the ignition voltage is

$$U_{ig} = \frac{B(pL)}{\ln\left[\dfrac{A}{\ln(1+1/\gamma)}\right] + \ln(pL)} \tag{6.29}$$

This function has a minimum for

$$(pL)_{min} = \frac{e}{A}\ln\left(1+\frac{1}{\gamma}\right) \tag{6.30}$$

which is

$$U_{min} = B(pL)_{min}$$

Problem 6.30. During the development of the dark discharge the electron number density exponentially increases from the cathode to the anode. This distribution gives rise to the diffusion flux of electrons from the anode to the cathode. Find the resulting effect on the development of the discharge.

If we neglect the escape of charged particles to the walls, the balance equation for the electron density can be written as

$$\frac{dN_e}{dt} = 0 = -\frac{dj}{dz} + \alpha w N_e$$

where w is the electron drift velocity. The electron flux is

$$j = wN_e - \mathscr{D}\frac{dN_e}{dz}$$

where $\mathscr{D}$ is the diffusion coefficient for electrons. Using the balance equation, we obtain

$$\mathscr{D}\frac{d^2N_e}{dz^2} - w\frac{dN_e}{dz} + \alpha w N_e = 0$$

The solution of this equation is $N_e = N_e(0)\exp(kz)$, where

$$k = \frac{w}{2\mathscr{D}} - \sqrt{\frac{w^2}{4\mathscr{D}^2}\left(1-\frac{4\alpha\mathscr{D}}{w}\right)} \approx \alpha\left(1-\frac{\alpha\mathscr{D}}{w}\right), \qquad \frac{\alpha\mathscr{D}}{w} \ll 1$$

and the condition of self-maintained discharge, according to eq. (6.26), is

$$kL = \ln\left(1+\frac{1}{\gamma}\right)$$

Let us estimate the parameter $\alpha\mathscr{D}/w$. Assume that electrons lose energy only for the excitation or ionization of atoms. Then we have $\alpha = \frac{eE}{I}\varkappa$, where I is the ionization potential of the atom, E is the electric field, and $\varkappa$ is the part of the electron energy lost in inelastic collisions, which is spent for ionization. Let us now make use of the relationships $w = KE$ and $\mathscr{D} \sim \frac{K}{e}\bar{\varepsilon}$, where K is the electron mobility and $\bar{\varepsilon}$ is the mean electron energy. Hence,

$$\frac{\alpha\mathscr{D}}{w} \sim \frac{\bar{\varepsilon}}{I}\varkappa \ll 1$$

since each of the factors is markedly smaller than unity.

Taking into account the above effect, we obtain the following condition for self-maintained discharge:

$$\alpha L\left(1 - \frac{\alpha\mathscr{D}}{w}\right) = \ln\left(1 + \frac{1}{\gamma}\right)$$

The factor in the parentheses on the left-hand side is close to unity.

Problem 6.31. Self-maintained discharge occurs in a cylindrical tube with the length L and the radius r_0 ($r_0 \ll L$). Find the condition for self-maintained discharge, taking into account the diffusion of electrons to the tube walls.

The equation for the electron number density $N_e(\mathbf{r}, t)$ in the discharge tube is

$$\frac{\partial N_e}{\partial t} = -\operatorname{div}\mathbf{j}_e + \nu N_e = 0$$

where the electron flux $\mathbf{j}_e$ is the sum of the diffusion flux $\mathscr{D}_e\nabla N_e$ and the hydrodynamic flux $\mathbf{w}_e N_e$, $\mathbf{w}_e$ is the electron drift velocity, $\mathscr{D}_e$ is the electron diffusion coefficient in the gas, and ν is the frequency of ionization of atoms, producing pairs of charged particles.

The boundary condition for this equation is $N_e(\rho = r_0) = 0$. A similar equation can be written for the number density of ions:

$$\frac{\partial N_i}{\partial t} = 0 = -\operatorname{div}\mathbf{j}_i + \nu N_e, \quad \mathbf{j}_i = -\mathscr{D}_i\nabla N_i + \mathbf{w}_i N_i$$

Here $\mathscr{D}_i$ is the diffusion coefficient for ions, and $\mathbf{w}_i$ is the ion drift velocity. The boundary conditions are: $N_i(\rho = r_0) = 0$ at the tube wall, at the cathode $\gamma w_i N_i = w_e N_e$, and $N_i(z = L) = 0$ at the anode.

Using the results of Problem 6.29, we shall neglect the diffusion fluxes of ions and electrons along the electric field (parallel to the tube axis).

Under the steady-state conditions we obtain the following system of equations:

$$\mathscr{D}_e \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dN_e}{d\rho}\right) - w_e\frac{dN_e}{dz} + \nu N_e = 0$$

$$\mathscr{D}_i \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dN_i}{d\rho}\right) + w_i\frac{dN_i}{dz} + \nu N_e = 0$$

$$N_e(\rho = r_0) = N_i(\rho = r_0) = 0, \quad N_i(z = L) = 0$$

$$\gamma w_i N_i(z = 0) = w_e N_e(z = 0)$$

Here the z-axis is parallel to the tube axis ($0 \leqslant z \leqslant L$), ρ is the distance to the tube axis, and we have taken into account the fact that the drift velocities of electrons and ions have opposite directions.

The variables in the above system of equations can be separated. We obtain

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dN_e}{d\rho}\right) + cN_e = 0$$

The solution of this equation is $N_e = \mathscr{I}_0(\sqrt{c}\,\rho)$, where $\mathscr{I}_0(x)$ is the Bessel function. From the condition $N_e(r_0) = 0$ we find the infinite set of discrete values of $\sqrt{c}$. However, the physical condition $N_e \gg 0$ for $\rho \leqslant r_0$ yields the only suitable solution $N_e = \mathscr{I}_0\left(2.405\frac{\rho}{r_0}\right) f_e(z)$. Now, using the balance equation for the electron number density, we obtain the equation for $f_e(z)$:

$$-w_e\frac{df_e}{dz} + \left(\nu - \frac{2.405^2}{r_0^2}\mathscr{D}_e\right)f_e = 0$$

The solution of this equation is

$$f_e = N_0 \exp(\alpha_1 z), \quad \text{where} \quad \alpha_1 = \alpha - \frac{\mathscr{D}_e}{w_e}\left(\frac{2.405}{r_0}\right)^2$$

and $\alpha = \nu/w_e$ is the first Townsend coefficient. As in the case of electrons, taking the ion number density in the form

$$N_i = f_i(z)\mathscr{I}_0\left(2.405\frac{\rho}{r_0}\right)$$

we obtain the following equation:

$$w_i\frac{df_i}{dz} - \left(\frac{2.405}{r_0}\right)^2 D_i f_i + \nu f_e = 0$$

The solution of this equation for the boundary condition $N_i(z = L) = 0$ is

$$N_i = \frac{w_e\alpha}{w_i(\alpha_1 - \delta)} \exp(\delta z) \{\exp[(\alpha_1 - \delta)L] - \exp[(\alpha_1 - \delta)z]\}$$

where

$$\delta = \frac{\mathscr{D}_i}{w_i}\left(\frac{2.405}{r_0}\right)^2$$

Using the boundary condition at the cathode, we obtain now the condition for the ignition of discharge:

$$\frac{\alpha}{\alpha'}[\exp(\alpha' L) - 1] = \frac{1}{\gamma}$$

Here

$$\alpha' = \alpha - \left(\frac{\mathscr{D}_e}{w_e} + \frac{\mathscr{D}_i}{w_i}\right)\left(\frac{2.405}{r_0}\right)^2$$

Let us consider the limiting cases. If for ions and electrons $\mathscr{D} \ll \frac{w}{r_0^2}L$, then $L(\alpha - \alpha') \ll 1$ and the loss of charged particles due to their escape to the wall can be neglected. Since $L\alpha \sim 1$, we have then $\alpha - \alpha' \ll \alpha$ and the condition of self-maintained discharge here is identical to the condition (6.25):

$$\exp(\alpha L) - 1 = 1/\gamma$$

In the other limiting case the loss of charged particles is due to their escape to the walls. Then we have $\alpha' \ll \alpha$, that is,

$$\alpha \approx \left(\frac{\mathscr{D}_e}{w_e} + \frac{\mathscr{D}_i}{w_i}\right)\left(\frac{2.405}{r_0}\right)^2$$

which is the condition of self-maintained discharge. If $\alpha' L \ll 1$, the condition is

$$\alpha L \gamma = 1$$

Problem 6.32. External radiation produces $1/\tau$ pairs of charged particles per unit time in a gas discharge tube. The time of ignition of discharge or the time of recombination of charged particles at the electrodes is short compared with τ. Find the probability that discharge ignites in the time t after the application to the tube of the potential difference U which is higher than the discharge ignition potential U_{ig}.

Denote by $a \leqslant 1$ the probability that the discharge ignites after the production of a pair of charged particles. The probability that discharge ignites per unit time is a/τ and the probability $W(t)$ that the charge ignites by the moment t can be found from the equation

$$\frac{dW}{dt}=\frac{a}{\tau}(1-W)$$

The equation yields

$$W(t)=1-\exp\left(-a\frac{t}{\tau}\right)$$

Let us find the probability a. The probability that the circuit is disconnected is

$$1-a=w_0'+w_1'w_0+w_1'w_1w_0+w_1'w_1^2w_0+\ldots=w_0'+\frac{w_1'w_0}{1-w_1}$$

where w_0 is the probability that the circuit is disconnected if one electron is produced at the cathode, w_1 is the probability that after a given production event involving one electron a new electron is produced at the cathode, w_0' and w_1' are the respective probabilities if one electron is produced in the discharge gap. When writing this equation,we have assumed that the production of two or more electrons at the cathode results in discharge ignition, since the probability of circuit disconnection in this case is very small.

The probability of circuit disconnection after the production of N ions in the discharge gap, that is, the probability that the collisions of these ions with the cathode do not produce free electrons, is $w_0=(1-\gamma)^N\approx$ $\approx\exp(-\gamma N)$, since $\gamma\ll 1$. The probability of the production of one electron at the cathode under the same conditions is $w_1=\gamma N(1-\gamma)^{N-1}\approx$ $\approx\gamma N\exp(-\gamma N)$. These probabilities should be averaged over the number of produced ions.

Since $\gamma\ll 1$, the number of produced ions should be large, $N\gg 1$. In this case the averaging over the Poisson distribution reduces to the replacement of the number N of ions by the mean number $\bar{N}$ of ions. If the initial electron was produced at the cathode, then $\bar{N}=\exp(\alpha L)$, if it was produced at the distance z from the cathode, then $\bar{N}=\exp(\alpha z)$. Averaging over z yields

$$\langle w_0'\rangle=\int_0^L\frac{dz}{L}\exp[-\gamma\exp(\alpha z)]=\frac{1}{\alpha L}[Ei(-\gamma\bar{N})-Ei(-\gamma)]$$

$$\langle w_1'\rangle=\int_0^L\frac{dz}{L}\gamma\exp(\alpha z)\exp[-\gamma\exp(\alpha z)]=\frac{\gamma}{\alpha L}\int_0^{\bar{N}}dN\exp(-\gamma N)=\frac{1-\exp(-\gamma\bar{N})}{\alpha L}$$

where $\bar{N} = \exp(\alpha L)$ and $Ei(x)$ is the exponential integral function. Hence, we obtain

$$1 - a = \frac{1}{\alpha L}\left[Ei(-\gamma\bar{N}) - Ei(-\gamma)\right] + \frac{[1 - \exp(-\gamma\bar{N})]\exp(-\gamma\bar{N})}{\alpha L\,[1 - \gamma\bar{N}\exp(-\gamma\bar{N})]}$$

where $\bar{N} = \exp(\alpha L)$ and α corresponds to the field strength U/L. In particular, if the potential at the electrodes is close to the ignition potential, then $\bar{N} - 1 = 1/\gamma$, $\gamma\bar{N} \approx 1$ and $(1 - \alpha)$ tends to 1 since $1 - \alpha = = \frac{1}{\alpha L}[\ln(1/\gamma) + 0.7] \approx 1$ owing to the fact that $\alpha L \approx \ln(1/\gamma)$ and $\gamma \approx 1$.

Problem 6.33. Gas discharge occurs between two coaxial cylindrical electrodes. The gas contains a small amount of an admixture with a low ionization potential. The pairs of charged particles are produced owing to the Penning effect after collisions of excited atoms with admixture atoms. Since the number density of admixture atoms is low, the excited atom travels a certain distance before colliding with the admixture atom, so that a pair of charged particles is produced at a certain distance from the point where the atom was excited. Assuming that this effect is weak, find how it affects the condition of self-maintenance of the discharge. The frequency of reaction between excited atoms and admixture atoms is $1/\tau$, and the diffusion coefficient of the excited atoms in the parent gas is $\mathcal{D}$.

By analogy with eq. (6.26), we can write the condition of self-maintenance of the discharge in the cylindrical coordinates in the following form:

$$\int_{\rho_1}^{\rho_2} \alpha\, dr = \ln\left(1 + \frac{1}{\gamma}\right)$$

Here ρ_1 is the distance from the symmetry axis to the smaller of the electrodes, and ρ_2 is the distance to the larger electrodes. The electric field in the discharge gap is

$$\mathbf{E} = \frac{\Delta U}{\ln(\rho_2/\rho_1)}\,\frac{\mathbf{n}}{\rho}$$

where ρ is the distance to the axis, $\mathbf{n}$ is the unit vector in the direction from a given point to the axis and perpendicular to it. The direction of the vector $\mathbf{E}$ does not depend on the sign of the potential difference ΔU between the electrodes, that is, on the polarity of the electrodes.

Since the electric field is parallel to $\boldsymbol{\rho}$, we shall consider only the displacements in this direction. If the excited atom is produced at the point ρ_0, the distribution along this direction owing to diffusion is

$$f(\rho,\, t) = (2\pi\mathscr{D}t)^{-1/2} \exp \frac{(\rho - \rho_0)^2}{4\mathscr{D}t}$$

Since the probability that a pair of charged particles is produced in the time interval from t to $t + dt$ is $\exp\,(-t/\tau)\; dt/\tau$, then the probability that a pair of charged particles is produced at the point ρ is

$$f(\rho) = \int\limits_0^{\infty} (2\pi\mathscr{D}T)^{-1/2} \exp \frac{(\rho - \rho_0)^2}{4\mathscr{D}t} \frac{\exp\,(-t/\tau)}{\tau}\, dt$$

The difference between the energies obtained from the field by electrons produced at the points ρ and ρ_0 is

$$\Delta\varepsilon = \int eE\,(\rho)\,(\rho - \rho_0) f(\rho)\, d\rho = e \left.\frac{dE}{d\rho}\right|_{\rho_0} \overline{(\rho - \rho_0)^2}$$

$$= e \left.\frac{dE}{d\rho}\right|_{\rho_0} \int\limits_0^{\infty} 2\mathscr{D}t \exp\,(-t/\tau)\, \frac{dt}{\tau} = 2\mathscr{D}\,\tau e \left.\frac{dE}{d\rho}\right|_{\rho_0}$$

By definition of the Townsend coefficient, the energy spent for the production of a pair of charged particles is $\varepsilon = eE/\alpha$ and, therefore, this effect results in the variation $\delta\alpha = \alpha \dfrac{\Delta\varepsilon}{\varepsilon} = \dfrac{\alpha^2 \Delta\varepsilon}{eE}$ of the Townsend coefficient in the system with cylindrical symmetry. Since $dE/d\rho = E/\rho$, we have

$$\delta\alpha = \frac{\alpha^2}{eE}\, 2\mathscr{D}\tau \frac{eE}{\rho_0} = \frac{2\mathscr{D}\tau}{\rho_0}\, \alpha^2$$

This is the difference in comparison with the plane system. We assume here that the energy ε spent for the production of one pair of charged particles and, hence, the Townsend coefficient do not vary significantly if the displacement ρ is of the order of $\mathscr{D}\tau$, that is, when the field varies by $\dfrac{E}{\rho} d\tau$. Taking into account the above consideration, we can write the condition of self-maintenance of the discharge as

$$\int\limits_{\rho_1}^{\rho_2} \alpha \left(1 \pm \frac{2D\tau\alpha}{\rho}\right) d\rho = \ln \left(1 + \frac{1}{\gamma}\right)$$

where the Townsend coefficient α is taken for the plane system. If the smaller electrode is the cathode, the plus sign is taken. In this case the effect en hances ionization, since the electrons produced nearer to the cathode gain from the field a greater energy than the energy lost by the electrons displaced towards the anode. If the larger electrode is the cathode, the minus sign is taken in the above condition.

Problem 6.34. Find the current-voltage dependence for the dark discharge between two plane electrodes.

Here we must take into account the presence of the space charge which affects the field in the discharge gap. Thus, the current of charged particles determining the space charge affects the potential difference between the electrodes. If the current is low, the charge density in the discharge gap is small and the electric field at each point is close to U/L, where U is the potenital difference between the electrodes, and L is the length of the discharge tube with the plane electrodes.

Write the first Townsend coefficient as

$$\alpha = \alpha\left(\frac{U}{L}\right) + \left(E - \frac{U}{L}\right)\alpha'$$

The difference between the field and U/L is due to the presence of the space charge. We can find the field from the Poisson equation

$$\frac{dE}{dz} = 4\pi e(N_i - N_e)$$

where z is the distance from the cathode, and N_i and N_e are the number densities of ions and electrons.

Since we consider the steady-state discharge, the electron current is equal to that of ions, and we have $w_e N_e = w_i N_i$, where w_e is the electron drift velocity, w_i is the ion drift velocity, the electron current is taken at the anode and the ion current is taken at the cathode. Since $w_e/w_i \sim \sqrt{M/m} \gg 1$ (where m and M are the masses of electrons and ions), we have $N_i \gg N_e$ in the discharge gap, that is, the electron number density is small compared with the ion number density and we can ignore it in the Poisson equation. Then, we can rewrite the Poisson equation as

$$\frac{dE}{dz} = 4\pi e N_i$$

The number densities of electrons and ions satisfy the following balance equations:

$$\frac{dN_e}{dt} = 0 = \frac{d}{dz}(w_e N_e) + \alpha w_e N_e$$

$$\frac{dN_i}{dt} = 0 = -\frac{d}{dz}(w_i N_i) + \alpha w_e N_e$$

The solution of these equations is

$$N_i = N_0 \frac{\exp(\alpha L) - \exp(\alpha z)}{\exp(\alpha L) - 1}$$

where N_0 is the ion number density at the cathode. Using eq. (6.26), we can express the ion number density as

$$N_i = N_0 \gamma\,[\exp(\alpha L) - \exp(\alpha z)] = N_0 \{1 + \gamma\,[1 - \exp(\alpha z)]\}$$

Substituting this expression into the Poisson equation and solving it, we obtain

$$E(z) = E(0) + 4\pi e N_0 \left[z + \gamma z - \frac{\gamma\,[\exp(\alpha z) - 1]}{\alpha} \right]$$

$$U(z) = E(0) z + 4\pi e N_0 \left\{ (1 + \gamma) \frac{z^2}{2} + \frac{\gamma}{\alpha} z - \frac{\gamma}{\alpha^2} [\exp(\alpha z) - 1] \right\}$$

and $U(0) = 0$. Since $\gamma \ll 1$ and $\alpha^2 L^2 \gg 1$, the potential difference between the electrodes is

$$U(L) = E(0) L + 2\pi e N_0 L^2$$

The condition of self-maintenance of the discharge is

$$\int_0^L \alpha\, dz = \ln\left(1 + \frac{1}{\gamma}\right)$$

For low currents the ignition potential of the discharge is $U_{ig} = E(0) L$. The Poisson equation yields $E(z) = U(L)/L + 4\pi e N_0 (z - L/2)$. Write down the potential difference between the electrodes as $U(L) = U_{ig} + \Delta U$. Use the following notation for $E = U_{ig}/L$: $\alpha_0 = \alpha$, $\alpha_0' = d\alpha/dE$, and $\alpha_0'' = d^2\alpha/dE^2$. The condition of self-maintenance of the discharge yields

$$\int_0^L \alpha\, dz = \alpha_0 L + \int_0^L \left[4\pi N_0 e \left(z - \frac{L}{2} \right) - \frac{\Delta U}{2} \right] \alpha_0'\, dz$$

$$+ \int_0^L \left[4\pi N_0 e \left(z - \frac{L}{2} \right) - \frac{\Delta U}{L} \right]^2 \frac{\alpha_0''}{2}\, dz = \ln\left(1 + \frac{1}{\gamma}\right)$$

Since the condition of self-maintenance of the discharge for low currents is $\alpha_0 L = \ln\ (1 + 1/\gamma)$, we obtain

$$-\alpha_0' \Delta U + \frac{2}{3}\pi^2 N_0^2 e^2 L^3 \alpha_0'' + \frac{\alpha_0''(\Delta U)^2}{2L} = 0$$

Since $\alpha_0'' \sim \alpha_0'/E \sim \alpha_0' L/U_{ig}$, the ratio of the last term to the first term in this expression is of the order of $\Delta U/U_{ig} \ll 1$. Ignoring the last term, we obtain $\Delta U = \dfrac{2}{3}\dfrac{\alpha_0''}{\alpha_0'}\pi^2 N_0^2 e^2 L^3$.

Now relate the potential drop to the discharge current. The total current is the sum of the ion current and the electron current which are identical owing to conservation of charge in the discharge gap. Therefore, the current density is $j = 2j_i = 2ew_i N_i$ if the sizes of the cathode and the anode are identical. Hence, we have $N_0 = j/2ew_i$ and

$$\Delta U = U_{ig} - U = \frac{\pi^2 \alpha_0''}{6\alpha_0'} j^2 L^3 \tag{6.31}$$

Typically, $\alpha_0' > 0$ and $\alpha_0'' > 0$ (for instance, when the Townsend coefficient is given by eq. (6.28) for weak fields). Therefore, the potential difference between the electrodes decreases with increasing current.

Problem 6.35. Compare the size of the discharge gap with the Debye-Hückel radius in the case of a dark discharge.

We shall ignore the energy losses by electrons in elastic collisions. Then the electron travelling in the electric field in a monoatomic gas increases its energy until it becomes of the order of the excitation energy of the atom and then spends its energy for excitation or ionization of an atom. Under these assumptions, the mean electron energy is of the order of the excitation energy of atoms. If the energy loss in elastic collisions is small, the first Townsend coefficient is $\alpha = eE\varkappa/I$, where I is the ionization potential of atoms, and $\varkappa < 1$ is the fraction of the electron energy lost in inelastic collisions, which is spent for atom ionization; by definition $1/\alpha$ is the distance passed by the electron, at which it produces a pair of charged particles.

Since $\alpha L \gtrsim 1$, we have the following relationship between the field and the size L of the discharge gap:

$$eEL \gtrsim I/\varkappa \gtrsim T/\varkappa$$

Here T is the mean electron energy. According to the definition of the dark discharge, we can neglect the space charge in it, that is in the Poisson equation $dE/dz = 4\pi e N_i$ the right-hand side is much smaller than E/L. Then we find $eE \gg 4\pi N_0 e^2 L$. Using the approximation $eE \gtrsim T/L$, we obtain further $L \ll r_D$, where $r_D = \sqrt{T/8\pi N_0 e^2}$ is the Debye-Hückel radius.

Thus, the condition of the dark discharge is that the Debye-Hückel radius for the discharge plasma must be much greater than the size of the discharge gap.

Problem 6.36. External radiation produces pairs of charged particles in the gas between two plane electrodes with the rate dN/dt per unit time per unit volume. The mean free paths of the charged particles are much smaller than the distance L between the electrodes. The potential difference between the electrodes is U. Find the potential distribution between the electrodes if the mobilities of electrons and ions in the gas are K_e and K_i and $K_e \gg K_i$.

The continuity equation for the electron number density in the steady-state case yields

$$\frac{dN}{dt} - \frac{dj_e}{e\,dx} = 0, \quad j_e = \frac{e\,dN}{dt}x$$

where x is the distance from the cathode. We have taken into account the fact that no charged particles are produced at the cathode, so that $j_e(0) = 0$. Similar considerations yield for the ion current density $j_i = \frac{dN}{dt}e(L - x)$. On the other hand, the electron current at the point x is

$$j_e(x) = eK_eE(x)N_e(x)$$

where E and N_e are the electric field and the electron number density at the point x. Using the above expression for j_e, we obtain for the number densities of electrons and ions at the point x

$$N_e(x) = \frac{\frac{dN}{dt}x}{K_eE(x)}, \quad N_i(x) = \frac{\frac{dN}{dt}(L - x)}{K_iE(x)}$$

Substitute these expressions into the Poisson equation:

$$\frac{dE}{dx} = 4\pi e\,(N_i - N_e) = \frac{4\pi e}{E}\frac{dN}{dt}\left(\frac{L - x}{K_i} - \frac{x}{K_e}\right)$$

Assuming that the mobilities of electrons and ions are independent of the field, we obtain

$$E^2 = E_0^2 + 8\pi e\frac{dN}{dt}\left(-\frac{x^2}{2K_e} + \frac{2Lx - x^2}{2K_i}\right)$$

Solving the equation $dU/dx = -E$ with the boundary conditions $U = 0$ for $x = L$ and $U = U_0$ for $x = 0$, we can find the potential distribution between the electrodes from this equation. The integration constant E_0 expressed in terms of U_0 and other parameters of the problem can be thus found.

Problem 6.37. The current density of the electrons emitted from the cathode is j. The potential difference between the cathode and the anode is U_0. Find the potential distribution between the cathode and the anode assuming that the mean free path of electrons is either much greater or much smaller than the gap size L. Assume that the electron mobility K is independent of the field in the second case.

The electron current density j is uniform in the gap since electrons are not produced or removed in the gap. Hence, we have

$$j = eN(x)v(x) = \text{const}$$

where $N(x)$ is the electron number density, and $v(x)$ is the electron velocity at the point x. If the mean free path of electrons is large and its collisions with the gas particles can be ignored, we have $v(x) = \sqrt{2Ue/m}$, where U is the potential at the point x (at the cathode we have $U(0) = 0$). Hence, $N(x) = j/e\sqrt{2Ue/m}$. Multiplying the Poisson equation $dE/dx = -4\pi Ne$ by $E = -dU/dx$, we obtain

$$\frac{1}{2}\frac{dE^2}{dx^2} = \frac{4\pi j\sqrt{m}}{\sqrt{2eU}}\frac{dU}{dx} = \frac{8\pi j\sqrt{m}}{\sqrt{2e}}\frac{dU^{1/2}}{dx}; \qquad E^2 = E_0^2 + \frac{8\pi j\sqrt{m}}{\sqrt{2eU}}$$

There are two types of boundary conditions for this equation, corresponding to two different situations. This problem deals with the processes in the diode vacuum tube. The boundary conditions $U = 0$ for $x = 0$ and $U = U_0$ for $x = L$ correspond to the case when the internal resistance of the source to which the diode is connected is much greater than the internal resistance of the diode. Under such conditions all the electrons emitted by the cathode get to the anode, and this situation is known as the filament saturation. The other case is realized when only a part of the electrons emitted from the cathode get to the anode. In this case the field at the cathode is not strong enough and a part of the emitted electrons return to the cathode, so that a cloud of electrons and ions is produced at the cathode, making up a high-conductivity region. Then the electric field at the cathode is zero, and the boundary condition is $-E = dU/dx|_{x=0} = 0$. The solution of the equation with the above boundary conditions is

$$U = \left(\frac{9\pi\sqrt{m}}{\sqrt{2e}}j\right)^{2/3} x^{4/3}$$

that is, the relationship between the potential difference U_0 and the current density is given by

$$U_0 = \left[\frac{9\pi\sqrt{m}}{\sqrt{2e}}j\right]^{2/3} L^{4/3}, \qquad j = \frac{2}{9\pi}\left(\frac{e}{2m}\right)^{1/2}\frac{U_0^{2/3}}{L^2}$$

This dependence is known as the *three-halves power law.*

If the mean free path of electrons is much smaller than the distance between the electrodes, the electron current density is $j = eKN(x)E(x)$, and from the Poisson equation $dE/dx = -4\pi eN(x)$ we obtain

$$E^2(x) = E_0^2 + \frac{8\pi j}{K}x, \quad U = \frac{K}{12\pi j}\left[\left(\frac{8\pi jx}{K} + E_0^2\right)^{3/2} - E_0^3\right]$$

We have used here the boundary condition $U = 0$ for $x = 0$.

If the electron current is much lower than the saturation current, the other boundary condition is $E = 0$ for $x = 0$. Then the solution of the equation is

$$U = \frac{4}{3}\sqrt{\frac{2\pi j}{eK}}\,x^{3/2}, \quad j = \frac{9eK}{32\pi}\frac{U_0^2}{L^3}$$

APPENDICES

A 1. CROSS SECTIONS OF COLLISIONS BETWEEN ATOMIC PARTICLES

Let us define the parameters describing a collision between two particles. The collision results in the changes in the directions of motion of the particles. If the internal states of the particles are not changed, the collision is called elastic, and if the internal state of one or both particles is changed, the collision is known as inelastic. Let us consider at first the inelastic collision, paying attention only to the internal states of the particles and disregarding their directions of motion.

Let a test particle be passed through a gas of another species. A collision of the test particle with a gas particle changes the internal state of the test particle, so that it goes over from the state 1 to the state 2. The character of the process is such that the probability $W_1(t)$ that the test particle is in the state 1 at the moment t satisfies the equation similar to the equation of radioactive decay:

$$\frac{dW_1}{dt} = -\nu_{12} W_1$$

Here ν_{12} is the frequency of transition of the test particle from the state 1 to the state 2, which describes the interaction between the test particle and the gas.

The inelastic transition occurs only after strong interaction between the test particle and the gas particles. Therefore, the transition frequency ν_{12} is proportional to the ratio between the volume around the gas particle, in which the strong interaction occurs, to the volume per one gas particle (naturally, this ratio is much smaller than unity). Since the volume per one gas particle is $1/N$, where N is the number density of gas particles, we have

$$\nu_{12} = N k_{12} \tag{A1.1}$$

where k_{12} is the rate constant for this process, which is independent of the number density of the gas particles and is, therefore, a parameter describing the elementary collision event. Another parameter which is more frequently used for describing elementary collision events in the theory of atomic collisions is the cross section of transition σ_{12} defined as

$$\sigma_{12} = \nu_{12}/Nv = k_{12}/v \tag{A1.2}$$

where v is the relative velocity of collision. Since Nv is the flux of the gas particles at the test particle, the cross section of the process is defined as the ratio of the transition frequency to the incident particle flux. Both the rate constant of the process and the transition cross section depend on the relative velocity of the colliding particles, so that both these parameters are equally useful for describing collisions between two particles.

The physical meaning of the transition cross section can be explained as follows. Let the test particle pass through a plane containing n gas particles per unit area. Then the probability that the test particle goes over from the state 1 to the state 2 is $n\sigma_{12} \ll 1$. Thus, σ_{12} can, indeed, be regarded as the effective cross section area around the gas particle, in which the test particle undergoes the inelastic transition.

Let us now apply the concept of collision cross section to the collisions in which the directions of motion of the colliding particles are changed. A convenient technique is to use the centre-of-mass reference system, since the motion of the centre of mass of the colliding particles is not affected by the collision. In the absence of external field this approach reduces the problem to that of the motion of a particle with the reduced mass in a central field. Thus, we can define the differential cross section of elastic or inelastic scattering in the centre-of-mass reference frame as the ratio of the number of particles scattered into the elementary solid angle per unit time to the incident particle flux.

Let us consider an elastic scattering of particles in the classical approximation, when the motion of particles is governed by classical laws. In the centre-of-mass reference frame this problem reduces to the analysis of the motion of one particle with the mass equal to the reduced mass of the colliding particles. The potential of interaction of this particle with the scattering centre is equal to the potential of interaction between the colliding particles. Fig. A1.1 illustrates the parameters describing elastic scattering of particles.

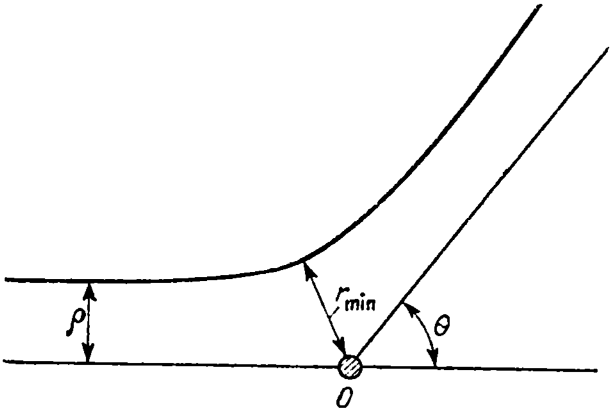

Fig. A1.1. The trajectory of relative motion of the colliding particles in the centre-of-mass reference frame. 0 — centre of mass. The trajectory is the totality of the points representing the vectors of the distance between the colliding particles. ρ is the impact parameter. θ is the scattering angle.

If the interaction potential is spherically symmetric, the scattering angle θ is determined by the impact parameter ρ of scattering, and if the dependence of the interaction potential on the distance between the particles is monotonic, the scattering angle is a single-valued function of the impact parameter. Therefore, there is a one-to-one relationship between the particles scattering into the angle element from θ to $\theta + d\theta$ and the impact parameter range from ρ to $\rho + d\rho$.

Let us derive the expression for the differential cross section of scattering. Assume that a particle flux Nv is incident on a scattering centre (N is the number density of particles and v is their velocity). The particles with the impact parameter in the range from ρ to $\rho + d\rho$ are scattered into a given solid angle element. The number of particles per unit time with the impact parameter in this range is equal to the number of particles in the volume $v \cdot 2\pi\rho \, d\rho$, and the number of particles scattered per unit time into the given solid angle element is $Nv \cdot 2\pi\rho \, d\rho$. Dividing this number by the incident particle flux, we obtain the differential cross section of scattering

$$d\sigma = 2\pi\rho \, d\rho \tag{A1.3}$$

This expression corresponds to the single-valued dependence of the scattering angle on the impact parameter. In the general case eq. (A1.3) must be replaced by the sum over the individual branches of the dependence $\theta(\rho)$.

Let us find the differential cross section of small-angle scattering. In this case the path of a particle in the central field is a straight line and its distance from the scattering centre given by the law of free motion is $R = \sqrt{\rho^2 + v^2t^2}$. The force with which the scattering centre acts on the particle in the direction perpendicular to the particle path is $\frac{\rho}{R}\left|\frac{\partial U}{\partial R}\right|$, so that the particle finally acquires the momentum in this direction equal to

$$\Delta p_\perp = \int\limits_{-\infty}^{+\infty} \frac{\rho}{R}\left|\frac{\partial U}{\partial R}\right| dt = \frac{2\rho}{v}\int\limits_{\rho}^{\infty}\left|\frac{\partial U}{\partial R}\right|\frac{dR}{\sqrt{R^2-\rho^2}}$$

Hence, the scattering angle corresponding to the impact parameter ρ is

$$\theta \approx \frac{\Delta p_\perp}{p} = \frac{\rho}{\varepsilon}\int\limits_{\rho}^{\infty}\left|\frac{\partial U}{\partial R}\right|\frac{dR}{\sqrt{R^2-\rho^2}} \tag{A1.4}$$

where $p = \mu v$ is the momentum in centre-of-mass reference frame, μ is the reduced mass of the colliding particles, and $\varepsilon = \mu v^2/2$ is the particle energy in the centre-of-mass reference frame.

In particular, if $U = C/R^n$, eq. (A1.4) yields $\theta = A/\varepsilon\rho^n$ and the differential cross section of scattering is

$$d\sigma = 2\pi\rho\, d\rho = \frac{2\pi}{n}\left(\frac{A}{\varepsilon}\right)^{2/n}\frac{d\theta}{\theta^{1+\frac{2}{n}}}, \quad \text{where} \quad A = \frac{C\sqrt{\pi}\,\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{2\Gamma\left(\frac{n}{2}+1\right)} \qquad \text{(A1.5)}$$

For the Coulomb interaction between particles $U = e^2/R$, and we obtain the Rutherford formula

$$d\sigma = \frac{2\pi A^2}{\varepsilon^2}\frac{d\theta}{\theta^3} = \frac{2\pi e^4}{\varepsilon^2}\frac{d\theta}{\theta^3} = \frac{\pi e^4\, d\Delta\varepsilon}{\varepsilon\Delta\varepsilon^2} \qquad \text{(A1.6)}$$

where $\Delta\varepsilon = \Delta p_\perp^2/2\mu = \varepsilon\theta^2$ is the energy exchange by the colliding particles in the centre-of-mass reference frame.

Let us now consider the integral cross sections of elastic scattering of particles. There are two types of them. One type is the large-angle scattering cross section. Cross sections of this type enter the equations for the transport coefficients, that is, the coefficients of diffusion, heat transfer, viscosity, mobility of particles, since the transport processes in the absence of inelastic scattering are determined by the large-angle scattering of particles. The cross sections of large-angle scattering are integrals of the differential cross sections, and the small scattering angles make small contributions to these parameters. For instance, the most widely used large-angle scattering cross section is the diffusion or transport cross section given by

$$\sigma^* = \int (1 - \cos\theta)\, d\sigma \qquad \text{(A1.7)}$$

It can be seen that small scattering angles appear in the integrand as $\theta^2/2$ and do not make a contribution to the integral.

Let us find a classical estimate of the cross section of large-angle scattering. The main contribution to the cross section is made by the scattering angles of the order of unity, and for these angles at the closest-approach distances between the particles the interaction is sufficiently strong, that is, the interaction potential at such distances is of the order of the kinetic energy of particles. Since the closest-approach distance in this case is of the order of the impact parameter, the cross section of large-angle scattering can be estimated as

$$\sigma = \pi R_0^2, \quad \text{where} \quad U(R_0) \sim \varepsilon \qquad \text{(A1.8)}$$

Here $U(R)$ is the potential of interaction between the colliding particles at the distance R between them, and ε is the energy of the particles in the centre-of-mass reference frame.

Another type of the integral cross section describing collisions between particles is the total cross section of scattering

$$\sigma_t = \int d\sigma \tag{A1.9}$$

The total cross section of scattering describes the variation of the phase of the radiating atom owing to its collisions with gas particles and, therefore, it appears in the expressions for the width of the spectral lines of radiation or absorption if the line broadening is due to collisions between particles.

It can be readily seen that in the classical limit the total cross section of scattering of particles is infinity. Indeed, in the classical approximation the scattering (even if very weak) occurs for any impact parameter, that is, the integral (A1.9) is infinity. Let us estimate the total cross section in the case when the motion of particles can be regarded as the motion of particles along classical paths. The classical approximation is not valid for large impact parameters for which the product of the angular momentum l by the scattering angle θ is of the order of unity. Cutting off integral (A1.9) at these impact parameters, we obtain the following expression for the total cross section of scattering:

$$\sigma_t = \pi\rho_t^2, \quad \text{where} \quad l(\rho_t)\,\theta(\rho_t) \sim 1$$

Here the angular momentum $l(\rho) = \mu\rho v/\hbar$, where μ is the reduced mass of the colliding particles, v is their relative velocity, and ρ is the impact parameter. According to eq. (A1.4), we have $\theta(\rho) \sim U(\rho)/\varepsilon \sim U(\rho)/\mu v^2$. Hence, we obtain the following estimate for the total cross section of scattering:

$$\sigma_t = \pi\rho_t^2, \quad \text{where} \quad \rho_t U(\rho_t)/\hbar v \sim 1 \tag{A1.10}$$

This estimate is valid if the motion of the colliding particles is governed by the classical laws, that is, if the main contribution to the cross section is made by large angular moments:

$$\mu v\rho_t/\hbar \gg 1 \tag{A1.11}$$

Let us compare the cross section of large-angle scattering and the total cross section of scattering, assuming that the motion of the particles is governed by the classical laws and the interaction potential steadily decreases with increasing distance between the particles. Eqs. (A1.9)-(A1.11) yield

$$U(R_0)/U(\rho_t) \sim \mu v\rho_t/\hbar \gg 1$$

that is, the total cross section of scattering is much greater than the cross section of large-angle scattering ($\sigma_t \gg \sigma$).

Finally, let us consider the scattering of particles in the model with the interaction potential

$$U = \begin{cases} 0 & \text{for } r > R_0 \\ \infty & \text{for } r \leqslant R_0 \end{cases} \tag{A1.12}$$

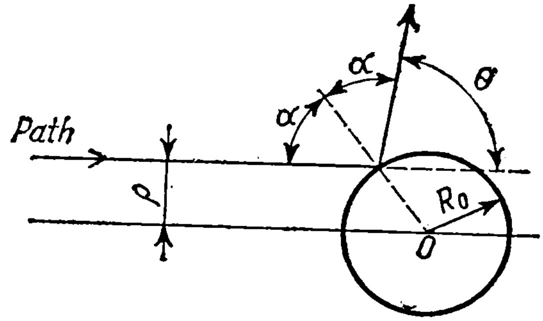

Fig. A1.2. Scattering of particles in the centre-of-mass reference frame for the rigid-sphere model.

This model, known as the *rigid-sphere model*, gives a good description of collisions between atoms or molecules and is widely used in physical kinetics of neutral gases. In the rigid-sphere model the scattering of particles in the centre-of-mass reference frame is similar to elastic scattering of a point by a solid spherical surface (see Fig. A1.2). Hence, we obtain $\theta = \pi - 2\alpha$, where $\sin\alpha = \rho/R_0$, that is, $\rho = R_0 \times \times \cos(\theta/2)$. Then the differential cross section of scattering is

$$d\sigma = 2\pi\rho\, d\rho = \frac{\pi R_0^2}{2} d(\cos\theta) \tag{A1.13}$$

and the diffusion cross section of scattering is

$$\sigma^* = \int (1 - \cos\theta)\, d\sigma = \pi R_0^2 \tag{A.1.14}$$

The cross section of large-angle scattering given by eq. (1.34), which enters the expressions for the heat conductivity and viscosity (see Problem 1.13 and 1.18), is

$$\sigma^2 = \int (1 - \cos^2\theta)\, d\sigma = \frac{2}{3}\pi R_0^2 \tag{A1.15}$$

We see that eqs. (A1.14) and (A1.15) agree with eq. (A1.8).

If the test particle is scattered by the gas particles, the frequency of scattering is $\nu \sim Nv\sigma$, where N is the number density of gas particles, v is the characteristic collision velocity, and σ is the characteristic cross section of large-angle scattering. Accordingly, we have $\lambda \sim v/\nu \sim (N\sigma)^{-1}$, where λ is the mean free path of the test particle in the gas, that is, the characteristic distance travelled by the test particle without colliding with the gas particles. It should be borne in mind that this is only an order-of-magnitude estimate of λ.

A 2. THE PRINCIPLE OF DETAILED BALANCING

The principle of detailed balancing relates the characteristics of an elementary process for the forward and reverse directions. This relationship corresponds to the symmetry of the system with respect to time reversal

Let us determine this relationship, proceeding from the microscopic description of the process.

Let the transition of the system from the state 1 to the state 2 or in the reverse direction be caused by the interaction described by the operator V. The probability of the transition per unit time is

$$w_{12} = \frac{2\pi}{\hbar} |V_{12}|^2 \frac{dg_2}{d\varepsilon} \tag{A2.1}$$

where V_{12} is the matrix element corresponding to the above states, and $dg_2/d\varepsilon$ is the statistical weight of the final state per unit energy. Let us consider a system of two particles whose collisions result in the transitions between the states of the system. Assume that before and after the collision the volume Ω contains one particle of each of the species. Then, by definition of the cross section σ_{12} of the transition between the states 1 and 2 and the cross section σ_{21} of the reverse transition, we obtain

$$\begin{aligned} w_{12} &= \frac{2\pi}{\hbar} |V_{12}|^2 \frac{dg_2}{d\varepsilon} = \frac{v_1 \sigma_{12}}{\Omega} \\ w_{21} &= \frac{2\pi}{\hbar} |V_{21}|^2 \frac{dg_1}{d\varepsilon} = \frac{v_2 \sigma_{21}}{\Omega} \end{aligned} \tag{A2.2}$$

where v_1 and v_2 are the relative collision velocities for the respective transitions.

The principle of detailed balancing corresponds to the invariance of the Hamiltonian under time reversal, that is, the matrix elements, for which the initial and final states are reversed, are identical, $|V_{12}| = |V_{21}|$. Hence, we obtain the following relationship between the cross sections of the forward and reverse processes:

$$v_1 \frac{dg_1}{d\varepsilon} \sigma_{12} = v_2 \frac{dg_2}{d\varepsilon} \sigma_{21} \tag{A2.3}$$

Let us consider specific cases. For elastic collision of two particles eq. (A2.3) yields

$$w_{12}\, dg_1 = w_{21} dg_2 \tag{A2.4}$$

Let us describe the state of each particle by its velocity. The velocities of the particles before the collision are $\mathbf{v}_1$ and $\mathbf{v}_2$ corresponding to the state 1 and the velocities after the collision are $\mathbf{v}_1'$ and $\mathbf{v}_2'$ corresponding to the state 2. The quantity $W(\mathbf{v}_1, \mathbf{v}_2 \to \mathbf{v}_1', \mathbf{v}_2')$ is the rate constant of the process in which the colliding particles have the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ and the scattered particles have the velocities $\mathbf{v}_1'$ and $\mathbf{v}_2'$. This quantity is proportional to the probability w_{12} of the transition per unit time for one particle of both species, and since the statistical weight of the continuous-spectrum states is proportional to the velocity range of the particles, eq. (A2.4) can be rewritten as

$$W(\mathbf{v}_1, \mathbf{v}_2 \to \mathbf{v}_1', \mathbf{v}_2')\, d\mathbf{v}_1\, d\mathbf{v}_2 = W(\mathbf{v}_1', \mathbf{v}_2' \to \mathbf{v}_1, \mathbf{v}_2)\, d\mathbf{v}_1'\, d\mathbf{v}_2' \qquad \text{(A2.5)}$$

Equation (A2.5) expresses the principle of detailed balancing for elastic collision of particles. If we take into account that $d\mathbf{v}_1\, d\mathbf{v}_2 = d\mathbf{v}_1'\, d\mathbf{v}_2'$, we obtain

$$W(\mathbf{v}_1, \mathbf{v}_2 \to \mathbf{v}_1', \mathbf{v}_2') = W(\mathbf{v}_1', \mathbf{v}_2' \to \mathbf{v}_1, \mathbf{v}_2) \qquad \text{(A2.6)}$$

For the inelastic collision

$$\mathrm{A} + \mathrm{B} \rightleftarrows \mathrm{A} + \mathrm{B}^* \qquad \text{(A2.7)}$$

the statistical weight of the initial state in eq. (A2.3) is

$$dg_1 = g_\mathrm{A} g_0 \frac{\Omega\, d\mathbf{p}_1}{(2\pi\hbar)^3}$$

where g_A is the statistical weight of the particle A, g_0 is the statistical weight of the particle B in the lower state, the third factor is the statistical weight of the continuous spectrum corresponding to the relative motion of the particles in the direct process, Ω is the fictitious volume containing the system of the colliding particles, and $\mathbf{p}_1$ is the momentum of the relative motion of the particles before the collision. The motion of the centre of mass is not considered. Using a similar expression for the statistical weight of the system after the collision, we can transform eq. (A2.3) for the process (A2.7) into

$$v_1\, d\mathbf{p}_1\, g_0 \sigma_\mathrm{exc}(v_1) = v_2\, d\mathbf{p}_2\, g_\mathrm{u} \sigma_\mathrm{q}(v_2)$$

where g_u is the statistical weight of the upper state of the particle B, σ_exc is the excitation cross section which is identical to σ_{12} in eq. (A2.3), and $\sigma_\mathrm{q} \equiv \sigma_{21}$ is the cross section of quenching.

Conservation of energy in the process (A2.7) yields

$$\frac{\mu v_1^2}{2} = \frac{\mu v_2^2}{2} + \Delta\varepsilon \qquad \text{(A2.8)}$$

where μ is the reduced mass of the colliding particles, v_1 and v_2 are their relative velocities before and after the collision, and $\Delta\varepsilon$ is the excitation energy. Hence, we have $v_1 dv_1 = v_2 dv_2$, and the above relationship between the cross sections of excitation and quenching is reduced to

$$g_0 v_1^2 \sigma_\mathrm{exc}(v_1) = g_\mathrm{u} v_2^2 \sigma_\mathrm{q}(v_2) \qquad \text{(A2.9)}$$

and the relative velocities of the particles before and after the collision are related by eq. (A2.8).

Let us analyze inelastic collisions of particles, in which the reduced mass is changed. An example of such processes is the dissociative recombination and the reverse process — the associative ionization:

$$e + AB^{+} \underset{\sigma_{\mathrm{ion}}}{\overset{\sigma_{\mathrm{rec}}}{\rightleftarrows}} A + B^{*} \tag{A2.10}$$

Here σ_{rec} is the cross section of dissociative recombination, and σ_{ion} is the cross section of associative ionization. Using the principle of detailed balancing (A2.3), we find the following relationship between these cross sections:

$$v_e g_e g_{AB^+} \frac{\Omega\, d\mathbf{p}_e}{(2\pi\hbar)^3} \sigma_{\mathrm{rec}} = v_{\mathrm{rel}} g_A g_B \frac{\Omega\, d\mathbf{p}_{\mathrm{rel}}}{(2\pi\hbar)^3} \sigma_{\mathrm{ion}}$$

Here g is the statistical weight of the respective particle, v_e and $\mathbf{p}_e$ is the velocity and momentum of the electron, v_{rel} is the relative velocity of the atoms, and $\mathbf{p}_{\mathrm{rel}}$ is the momentum of the relative motion of the atoms. Conservation of energy for the process yields

$$\frac{mv_e^2}{2} = \frac{\mu v_{\mathrm{rel}}^2}{2} + \Delta\varepsilon$$

where m is the electron mass, μ is the reduced mass of the nuclei, and $\Delta\varepsilon$ is the variation of the internal energy in the process. Since $v_e dp_e = v_{\mathrm{rel}} dp_{\mathrm{rel}}$, we obtain now

$$m^2 v_e^2 g_e g_{AB^+} \sigma_{\mathrm{rec}}(v_e) = \mu^2 v_{\mathrm{rel}}^2 g_A g_B \sigma_{\mathrm{ion}}(v_{\mathrm{rel}}) \tag{A2.11}$$

The reduced mass of the colliding particles varies also in radiative recombination of two particles and the reverse process — photodissociation of the bound state. A typical example is the process

$$e + A^{+} \underset{\sigma_{\mathrm{ion}}}{\overset{\sigma_{\mathrm{rec}}}{\rightleftarrows}} A + h\nu \tag{A2.12}$$

where σ_{rec} is the cross section of photorecombination, and σ_{ion} is the cross section of photoionization. For these processes eq. (A2.3) and conservation of energy yield

$$v_e g_e g_{A^+} \frac{\Omega\, d\mathbf{p}_e}{(2\pi\hbar)^3} \sigma_{\mathrm{rec}} = c g_A \frac{\Omega \cdot 2\, d\mathbf{k}}{(2\pi)^3} \sigma_{\mathrm{ion}}, \quad \frac{mv_e^2}{2} + J = \hbar\omega$$

where g_e, g_{A^+}, g_A and 2 are the statistical weights of the electron, ion, atom and photon, c is the velocity of light, J is the ionization potential of the atom, $\hbar\omega$ is the photon energy, and $\mathbf{k}$ is the wave vector of the photon, which is related to the frequency ω by the dispersion relation $\omega = kc$.

Other symbols have been described above. Hence, we obtain

$$\frac{m^2 v_e^2}{\hbar^2} g_e g_{A^+} \sigma_{rec} = 2 g_A k^2 \sigma_{ion}$$

Using the wave vector of the electron, $q = mv_e/\hbar$, and taking into account that $g_e = 2$ we find

$$\sigma_{rec} = \frac{g_A}{g_{A^+}} \frac{k^2}{q^2} \sigma_{ion} \tag{A2.13}$$

In conclusion, let us apply the principle of detailed balancing to the processes in which the number of particles is changed. An important process of this type is the process of formation of the bound state in three-particle collisions and the reverse process of dissociation of the bound state:

$$A + B + M \underset{k_{dis}}{\overset{\mathscr{K}}{\rightleftarrows}} AB + M \tag{A2.14}$$

Here the particle M carries off the excess energy in the direct process, $\mathscr{K}$ is the rate constant of the three-body process, and k_{dis} is the rate constant of dissociation of the bound state in two-particle collision. The balance equation for the number density of the particles AB in the given state can be derived from the reaction equation (A2.14):

$$\frac{d[AB]}{dt} = \mathscr{K}[A]\,[B]\,[M] - k_{dis}[AB]\,[M]$$

At equilibrium the right-hand side of this equation is zero. For instance, at thermodynamic equilibrium at the temperature T we have

$$\mathscr{K}(T)\,[A]_0\,[B]_0 = k_{dis}[AB]_0$$

where the subscript 0 denotes the number densities of particles at thermodynamic equilibrium.

The quantity

$$K_{equil}(T) \equiv \frac{[A]_0[B]_0}{[AB]_0}$$

which depends only on the gas temperature, is known as the equilibrium constant. (For instance, according to the Saha distribution (2.14), the equilibrium constant for the process $e + A^+ \rightleftarrows A$ is

$$K_{equil}(T) = \frac{g_e g_{A^+}}{g_A} \left(\frac{mT}{2\pi\hbar^2}\right)^{3/2} \exp\,(-J/T)$$

where g is the statistical weight of the respective particle, and J is the ionization potential of the atom A.) It can be seen that the rate constants of the forward and reverse processes are related by

$$k_{\mathrm{dis}}(T) = \mathcal{K}(T)\, K_{\mathrm{equil}}(T) \qquad \text{(A2.15)}$$

Though this expression has been derived under the assumption of thermodynamic equilibrium in the system, it is valid in the absence of thermodynamic equilibrium in the process (A2.14). The assumption about thermodynamic equilibrium was made only for derivation of the relationship between the rate constants. Therefore, eq. (A2.15) is valid for any ratio between the number densities of particles in the free and bound states, assuming that the velocity distribution of the colliding particles has the Maxwellian form corresponding to the temperature T.

A 3. THE CONDITIONS OF GASEOUSNESS OF A SYSTEM OF PARTICLES AND PLASMA IDEALITY

A system of particles is a gas if each particle of the system is free most of the time. In this case the potential of interaction between two particles, if the distance between them is of the order of the mean distance $N^{-1/3}$ between the neighbouring particles (N is the number density of particles), is much lower than the mean kinetic energy of the particles, which is of the order of the thermal energy T. Thus, the criterion of the gaseousness of a system of particles is

$$U(N^{-1/3}) \ll T \qquad \text{(A3.1)}$$

where $U(R)$ is the potential of interaction between two particles of the system for the distance R between them.

Let us now use another approach to determine the criterion of gaseousness. Define the free path of a test particle in the gas as

$$\lambda \sim 1/N\sigma \qquad \text{(A3.2)}$$

where σ is the cross section of large-angle scattering for collisions between the test particle and the gas particles. The free path of the particle is the characteristic distance which the test particle travels between two successive collisions with the gas particles. Hence, the gas can be defined as a system of particles in which the free path of a particle is much greater than the characteristic distance R_0 between the particles, corresponding to strong interaction between them:

$$\lambda \gg R_0, \quad \text{where} \quad U(R_0) \sim T$$

Using the definitions of the cross section of large-angle scattering (A1.8) and the free path (A3.2), we obtain the following criterion of gaseousness for a system of particles:

$$N\sigma^{3/2} \ll 1 \qquad \text{(A3.3)}$$

Here σ is the cross section of large-angle scattering of two particles for the collision energy of the order of the thermal energy T of the particles.

Compare the criterions (A3.1) and (A3.3). Assume that the potential $U(R)$ of the pairwise interaction between the particles steadily decreases with increasing distance R between them. The definition of the cross section (A1.8) of large-angle scattering yields $U(\sqrt{\sigma}) \sim T$. If we compare this relation and eq. (A3.1), the latter shows that the distance $\sqrt{\sigma}$ is much smaller than the mean distance $N^{-1/3}$ between the particles. This agrees with the criterion (A3.3), and we see that the criteria of gaseousness given by eqs. (A3.1) and (A3.3) are equivalent.

We consider plasma to be ideal if the interaction between charged particles satisfies the criterion of gaseousness. Assume that the plasma consists of singly charged ions and electrons. The Coulomb interaction potential for the charged particles is $|U(R)| = e^2/R$, where e is the electron charge and R is the distance between the particles. Substituting this interaction potential into the criteria (A3.1) and (A3.3), we obtain the following condition of plasma ideality:

$$N_e e^6/T^3 \ll 1 \qquad \text{(A3.4)}$$

where N_e is the number density of charged particles. Equations (A3.1) and (A3.3) both yield the condition (A3.4).

A 4. BROADENING OF SPECTRAL LINES

Let a gas consist of atoms or molecules in which a radiative transition can occur between two states with the energy difference $\hbar\omega_0$. The frequency distribution function a_ω for the emitted photons is defined so that $a_\omega\, d\omega$ is the probability of emission of the photon with the frequency in the range from ω to $\omega + d\omega$. For instance, in the case of radiative transition between two states the distribution function is $\delta(\omega - \omega_0)$, that is, only the photons with the energy $\hbar\omega_0$ are emitted. The same is true for the absorption of radiation in the given gas, that is, only the photons with the energy $\hbar\omega_0$ are absorbed owing to this transition.

In fact, emission and absorption of radiation due to this transition occur in a certain frequency range around ω_0 owing to various processes occurring in the gas. The frequency distribution of the emitted or absorbed photons described by the function a_ω determines the absorption and propagation of photons in the gas. Now, we have to identify the typical mechanisms of spectral line broadening and find the frequency distributions of photons for each of them. We shall consider radiation of moderate intensities, so that frequency distributions of photons will be independent of the radiation intensity.

The simplest mechanism of line broadening is due to the motion of atoms and known as the Doppler broadening. According to the Doppler law, a photon with the frequency ω_0 emitted by an atom approaching the radiation detector with the velocity v_x is detected as having the frequency $\omega = \omega_0(1 + v_x/c)$, where c is the velocity of light. Therefore, radiation of a system of atoms is spread over a certain frequency range determined by the velocity distribution of the atoms.

Let $f(v_x)\,dv_x$ be the probability that the atom's velocity along the x-axis is in the range from v_x to $v_x + dv_x$, so that $\int\limits_{-\infty}^{+\infty} f(v_x)\,dv_x = 1$. Since the frequency of emitted photons is determined by the atom velocities, we have

$$a_\omega\,d\omega = f(v_x)\,dv_x$$

and, hence, $a_\omega = \frac{c}{\omega_0} f\left[\frac{c}{\omega_0}(\omega - \omega_0)\right]$. For instance, for the Maxwellian velocity distribution we obtain

$$a_\omega = \frac{1}{\omega_0}\sqrt{\frac{Mc^2}{2\pi T}}\exp\left[-\frac{Mc^2}{2T}\frac{(\omega-\omega_0)^2}{\omega_0^2}\right] \tag{A4.1}$$

where M is the mass of the emitting atom.

We see that the characteristic width $\Delta\omega \sim \omega_0\sqrt{T/Mc^2}$ of the spectral line in this case is small compared with the frequency of the emitted photon since the thermal velocity of atoms is much lower than the velocity of light. This mechanism of line broadening is known as the *Doppler broadening*.

Other mechanisms of broadening are due to the interaction of the emitting or absorbing atomic particle with the surrounding gas particles. First, let us consider the case when the time of the strong interaction of the emitting atomic particle with the surrounding gas particles is short compared with the characteristic time of broadening. This is the case if the emitting particle occasionally collides with a perturbing particle and the time of collision is much shorter than the time between two successive collisions. This mechanism of broadening is known as the *impact broadening*.

Denote by $\varphi(t)$ the amplitude of the electromagnetic wave, that is, the strength of the electric or magnetic field of the wave. The Fourier component φ_ω of the field amplitude describes the spectral composition of the radiation, and the frequency distribution of photon is a function of the squared field amplitude, $a_\omega \sim |\varphi_\omega|^2$.

Let us consider the simple case when a transition is accompanied by weak decay owing to the finite lifetimes of the states between which the transition occurs. Then, the electromagnetic field amplitude is

$$\varphi(t) \sim \exp(i\omega_0 t - \nu t)$$

where ω_0 is the frequency of the transition, and $\nu \ll \omega_0$ describes the decay. Hence, we have $\varphi_\omega \sim [\nu + i(\omega - \omega_0)]^{-1}$ and $a_\omega \sim [\nu^2 + (\omega - \omega_0)^2]^{-1}$. The normalization condition $\int a_\omega \, d\omega = 1$ for the frequency distribution of photons yields

$$a_\omega = \frac{\nu}{\pi[\nu^2 + (\omega - \omega_0)^2]} \tag{A4.2}$$

This line shape is known as the *Lorentz shape.*

Let us return to the impact line broadening. In this case the atomic particle most of the time behaves as a free particle, and the amplitude of the emitted electromagnetic wave is $\exp(i\omega_0 t)$. During collision the frequency of the field is shifted, but since the duration of collision is short, it ultimately results in a shift of the wave phase. Thus, the wave amplitude can be written as

$$\varphi(t) \sim \exp\left[i\omega_0 t + i\sum_k \chi_k \eta(t - t_k)\right]$$

where $\eta(x)$ is the unit function, that is $\eta(x) = 0$ for $x < 0$ and $\eta(x) = 1$ for $x > 0$, χ_k is the phase shift owing to the kth collision, and t_k is the time of the kth collision. The Fourier component of this expression is

$$\varphi_\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(t) \exp(-i\omega t)\, dt \sim \sum_k \frac{[1 - \exp(i\omega_0 \tau_k)]}{(\omega - \omega_0)} \exp\left[i\left(\sum_{j<k} \chi_j + \frac{\pi}{2}\right)\right]$$

where τ_k is the time interval between the kth and the $(k + 1)$th collision.

To find the frequency distribution $a_\omega \sim |\varphi_\omega|^2$ of photons, average it over the phase shifts χ_k. Assume that χ_k are random and not small, so that the mean value $\overline{\exp[i(\chi_k - \chi_j)]} = \delta_{jk}$, that is, it is 0 for $j \neq k$ and 1 for $j = k$. Since $|1 - \exp(ix)|^2 = 2(1 - \cos x)$, we obtain

$$a_\omega = \frac{C}{(\omega - \omega_0)^2} \langle 1 - \cos(\omega - \omega_0)t \rangle \tag{A4.3}$$

where C is the normalization factor, t is the time interval between two successive collisions, and angle brackets denote the averaging over t. Let us perform this averaging.

Denote by $W(t)$ the probability that the first collision of the emitting particle with surrounding particles occurs at the time t after the starting moment. By definition, this function satisfies the equation $dW/dt = -\nu W$, where ν is the collision frequency and, according to the initial condition $W(0) = 1$, we obtain $W(t) = \exp(-\nu t)$. Hence, the probability that the collision occurs in the time interval from t to $t + dt$ is $\left|\frac{dW}{dt}\right| dt = \exp(-\nu t)\,\nu\,dt$. Now we can find the mean time interval between two successive collisions:

$$\bar{t} = \int_0^\infty t \left|\frac{dW}{dt}\right| dt = \frac{1}{\nu}$$

Using this expression and the normalization condition for the frequency distribution of emitted photons, we can rewrite eq. (A4.3) as

$$a_\omega = \frac{\nu}{\pi[\nu^2 + (\omega - \omega_0)^2]} \tag{A4.4}$$

Thus, the impact broadening results in the Lorentz line shape as in the case of broadening due to damping, described by eq. (A4.2) which is identical to eq. (A4.4) but for the definition of ν.

The parameter ν in eq. (A4.4) by the order of magnitude is

$$\nu \sim Nv\sigma \tag{A4.5}$$

where N is the number density of the gas particles, v is the characteristic collision velocity, and σ is the collision cross section. We shall assume that impact broadening of spectral lines is determined only by the upper state of the emitting atomic particle.

Let us now identify the type of collision cross section entering eq. (A4.5), assuming that the motion of the colliding particles can be described by classical equations. The nature of impact broadening shows that the main contribution to the cross section is made by the impact parameters for which the phase shift χ is of the order of unity. Hence, $\sigma \sim \rho_{\text{eff}}^2$ and for the impact parameter ρ_{eff} we have

$$\int \frac{U(R)\,dt}{\hbar} \sim 1$$

where $U(R)$ is the potential of interaction between the perturbing particle and the emitting particle in the upper state. Since the particle's path for this impact parameter is a straight line, we have

$$\int \frac{U(R)}{\hbar} dt \sim \frac{\rho U(R)}{\hbar v} \tag{A4.6}$$

that is, the cross section by the order of the magnitude is

$$\sigma \sim \rho_{\text{eff}}^2, \quad \text{where} \quad \rho_{\text{eff}} U(\rho_{\text{eff}})/\hbar v \sim 1$$

Thus, the cross section entering eq. (A4.5) is identical to the total cross section (A1.10) for collisions between the emitting atomic particle and a gas particle.

Let us derive the condition of applicability of impact broadening of spectral lines. The time in which the particle passes through a region with the size ρ_{eff} where scattering occurs must be much shorter than the time interval between two successive collisions, that is,

$$\rho_{\text{eff}}/v \ll (Nv\sigma_t)^{-1}$$

where σ_t is the total cross section of the collision. Then we have

$$N\sigma_t^{3/2} \ll 1 \tag{A4.7}$$

which is the condition of the applicability of the impact broadening mechanism.

Let us now analyze the other limiting case of impact broadening, assuming that the characteristic times of broadening is so short that the perturbing particles do not have time to travel a noticeable distance. Then the perturbing particles can be assumed to be stationary and the apparent shift of the spectral line is given by the relationship

$$\omega - \omega_0 = \frac{1}{\hbar} \sum_i V(\mathbf{R}_i), \tag{A4.8}$$

where V is the difference between the interaction potentials for the perturbing particle and the emitting particle in the upper and lower states, and $\mathbf{R}_i$ is the coordinate of the ith particle if the origin of the coordinates is the nucleus of the emitting particle.

Equation (A4.8) gives the shift of the spectral line for a given configuration of the perturbing particles. To find the frequency distribution a_ω

of the emitted photons, we have to average eq. (A4.8) over these configurations. Since the interaction of particles in a gas has the binary nature, we can, using eq. (A4.8), readily estimate the characteristic shifts and widths of spectral lines, which are of the same order of magnitude. These quantities amount to the frequency shift owing to the pairwise interaction between the emitting particle and the gas particle for the mean distance between them:

$$\nu \sim \frac{1}{\hbar} V(N^{-1/3}) \tag{A4.9}$$

Since the probability that the distance between the perturbing particle and the emitting particle is in the range from R and $R + dR$ is $4\pi R^2 \, dR \, N$, the frequency distribution of photons at the spectral line wing is

$$a_\omega \, d\omega = 4\pi R^2 \, dR \, N \tag{A4.10a}$$

The line shift is

$$\omega - \omega_0 = \frac{1}{\hbar} V(R) \tag{A4.10b}$$

For instance, if $V(R) = CR^{-n}$, eqs. (A4.10) yield the following frequency distribution of photons at the line wing:

$$a_\omega \, d\omega = \frac{4\pi N C^{\frac{3}{n}}}{n} \frac{d\omega}{(\omega - \omega_0)^{1+\frac{3}{n}}} \tag{A4.11}$$

Let us derive the criterion of applicability of the static theory of line broadening. The configuration of the particles must remain unchanged during the characteristic times of broadening. In particular, for the main photon frequency range the condition is that the displacement of the particles during this time (which is of the order of $1/\nu$) must be much smaller than the mean distance between the particles:

$$vN^{1/3} \ll \nu \sim \frac{1}{\hbar} V(N^{-1/3}) \tag{A4.12}$$

Compare this criterion with the criterion of applicability of the impact broadening theory. Consider the case when broadening is due to the interaction only with the excited state of the emitting atomic particle, so that V

in eq. (A4.12) coincides with the interaction potential U in eq. (A4.6) for the excited state of the emitting particle and the gas particle. Write down the criterion of applicability of the static broadening theory given by eq. (A4.12) as $N^{-1/3}U(N^{-1/3})/\hbar v \gg 1$ and compare it with eqs. (A1.10) and (A4.8). Hence, for the interaction potential steadily decreasing with distance we obtain $\sqrt{\sigma_t} \gg N^{-1/3}$. Hence, the criterion of applicability of the static theory of broadening can be expressed as

$$N\sigma_t^{3/2} \gg 1 \tag{A4.13}$$

Comparison of this expression with eq. (A4.7) shows that the impact theory and the static theory of line broadening describe diametrically opposite cases of broadening due to the interaction between the emitting particle and the surrounding gas particles.

A 5. SOME FUNDAMENTAL PHYSICAL QUANTITIES AND RELATIONSHIPS USED IN PLASMA PHYSICS

1) PHYSICAL CONSTANTS

Velocity of light	$c = 2.99792 \times 10^{10}$ cm/s	Electron mass	9.1095×10^{-28} g
Planck constant	$h = 6.62619 \times 10^{-27}$ erg·s;	Proton mass	1.6726×10^{-24} g
	$\hbar = 1.05459 \times 10^{-27}$ erg·s	Atomic mass unit (amu)	1.6605×10^{-24} g
Electron charge	$e = 4.8032 \times 10^{-10}$ CGSE units;	Avogadro number	6.0221×10^{23} mole^{-1}
	$e^2 = 2.3071 \times 10^{-19}$ CGSE units	Stefan-Boltzmann constant	$\sigma = 5.669 \times 10^{-12}$ W/cm^2·K^4

2) CONVERSION FACTORS FOR THE ENERGY UNITS

	erg	eV	cal/mole	cm^{-1}	K	rydberg
erg	1	6.2418×10^{11}	1.4395×10^{16}	5.0346×10^{15}	7.2432×10^{15}	4.5878×10^{10}
eV	1.6021×10^{-12}	1	2.3063×10^{4}	8.0660×10^{3}	1.1605×10^{4}	7.3501×10^{-2}
cal/mole	6.9466×10^{-17}	4.3359×10^{-5}	1	0.34973	0.50319	3.1870×10^{-6}
cm^{-1}	1.9862×10^{-16}	1.2398×10^{-4}	2.8592	1	1.4386	9.1127×10^{-6}
K	1.3806×10^{-16}	8.6170×10^{-5}	1.9873	0.69509	1	6.3344×10^{-6}
rydberg	2.1797×10^{-11}	13.605	3.1377×10^{5}	1.0973×10^{5}	1.5787×10^{5}	1

3) SIMPLE RELATIONS IN TERMS OF CONVENIENT UNITS

Relation	Format	C	Units
1. $v = \sqrt{2\varepsilon/M}$ v — velocity; ε — energy; M — particle mass	$v = C\sqrt{\varepsilon/M}$ (cm/s)	5.931×10^7 1.389×10^6 5.506×10^5	ε in eV, M in electron mass units ε in eV, M in amu ε in K, M in electron mass units
2. $\omega = \varepsilon/\hbar$ ε — photon energy; ω — frequency	$\omega = C\varepsilon$ (s^{-1})	1.289×10^4 1.520×10^{15}	ε in K, M in amu ε in eV
3. $\omega = c/2\pi\lambda$ ω — frequency; λ — wavelength of electromagnetic wave; c — velocity of light	$\omega = C\lambda$ (s^{-1})	1.885×10^{11}	λ in cm
4. $\omega_0 = (4\pi Ne^2/m)^{1/2}$ ω_0 — electron plasma frequency; N — electron number density; e — electron charge; m — electron mass	$\omega_0 = C\sqrt{N}$ (s^{-1})	5.642×10^4	N in cm^{-3}
5. $r_D = (T/8\pi Ne^2)^{1/2}$ r_D — Debye-Hückel radius; N — number density of charged particles; T — plasma temperature; e — electron charge	$r_D = C\sqrt{T/N}$ (cm)	525.6 4.879	N in cm^{-3}, T in eV N in cm^{-3}, T in K
6. $\omega_H = eH/mc$ ω_H — Larmor frequency; e — electron charge; m — electron mass; c — velocity of light; H — magnetic field	$\omega_H = CH$ (s^{-1})	1.759×10^7	H in Gs

7. $\omega_H = eH/Mc$ — ion Larmor frequency; M — ion mass	$\omega_H = C \dfrac{H}{M} (\mathrm{s}^{-1})$	9649	H in Gs, M in amu
8. $r_H = v/\omega_H$ — electron Larmor radius; v — electron velocity; ω_H — electron Larmor frequency	$r_H = C \dfrac{\sqrt{\varepsilon}}{H} (\mathrm{cm})$	3.372	ε in eV, H in Gs
9. $r_H = v/\omega_H$ — ion Larmor radius; v — ion velocity; ω_H — ion Larmor frequency	$r_H = C \dfrac{\sqrt{\varepsilon M}}{H} (\mathrm{cm})$	144	ε in eV, M in amu, H in Gs
10. $N = p/T$ — number density of gas particles; p — gas pressure; T — temperature	$N = C \dfrac{p}{T} (\mathrm{cm}^{-3})$	7.340×10^{21} 9.658×10^{18}	p in atm; T in K, p in torr, T in K
11. $N_e e^6/T^3$ — gas ideality parameter; N_e — electron number density; e — electron charge; T — plasma temperature	$C \dfrac{N_e}{T^3}$	2.986×10^{-21} 4.668×10^{-9}	N_e in cm^{-3}, T in eV N_e in cm^{-3}, T in K
12. $\mathscr{D}N = \dfrac{3\sqrt{\pi T}}{8\sqrt{2\mu}\,\bar{\sigma}}$	$\mathscr{D}N = C \dfrac{\sqrt{T}}{\sqrt{\mu}\,\bar{\sigma}} (\mathrm{cm}^{-1}\,\mathrm{s}^{-1})$	4.278×10^{19}	T in K, μ in amu, $\bar{\sigma}$ in Å^2 (10^{-16} cm^2)
$\mathscr{D}$ — diffiusion coeficient in Chapman-Enskog approximation; N — gas density; T — gas temperature; μ — reduced mass; $\bar{\sigma}$ — averaged collision cross section	$\mathscr{D} = \dfrac{C\sqrt{T}}{\sqrt{\mu}\,\bar{\sigma}} (\mathrm{cm}^2/\mathrm{s})$	1.595	*Ditto.* $N = 2.687 \times 10^{19}$ cm^{-3} (at 1 atm and 0°C)

Continued

Relation	Format	C	Units
13. Einstein relation $K = e\mathscr{D}/T$ K – mobility; $\mathscr{D}$ – diffusion coefficient; e – ion charge; T – gas temperature	$K = C\dfrac{\mathscr{D}}{T}$ (cm^2/V·s)	1.045×10^9	$\mathscr{D}$ in cm^2/s, T in K
14. $\eta = 5\sqrt{\pi TM}/24\bar{\sigma}_2$ – gas viscosity in Chapman-Enskog approximation; T – temperature; M – particle mass; $\bar{\sigma}_2$ – averaged cross section	$\eta = C\dfrac{\sqrt{TM}}{\bar{\sigma}_2}$ (g/cm·s)	5.591×10^{-5}	T in K, M in amu, $\bar{\sigma}_2$ in Å^2 (10^{-16} cm^2)
15. $\varkappa = 25\sqrt{\pi T}/32\sqrt{M}\,\bar{\sigma}_2$ – heat conductivity in Chapman-Enskog approximation. Notation as above	$\varkappa = C\dfrac{\sqrt{T}}{\sqrt{M}\,\bar{\sigma}_2}$ (W/cm·K)	1.743×10^{-8}	*Ditto*
16. $K = 3\sqrt{\pi}e/8N\sqrt{2T\mu}\,\bar{\sigma}$ – ion mobility in Chapman-Enskog approximation e – ion charge; N – gas density; T – temperature; μ – reduced mass of ion and gas molecule; $\bar{\sigma}$ – collision cross section	$K = C/\sqrt{T\mu}N\bar{\sigma}$ (cm^2/V·s)	1.850×10^4	T in K, μ in amu, $\bar{\sigma}$ in Å^2 (10^{-16} cm^2), N at 1 atm and 0°C
17. $eE/TN\sigma$ e – electron charge; E – electric field; T – gas temperature; N – gas density; $\bar{\sigma}$ – collision cross section	$C = E/TN\sigma$	1.160×10^{20} 4.316	E in V/cm, σ in Å^2, T in K, N in cm^{-3} *Ditto*. N at 1 atm and 0°C

A 6. ATOMIC AND MOLECULAR PARAMETERS

1) IONIZATION POTENTIALS FOR ATOMS AND THEIR SINGLY AND DOUBLY CHARGED IONS (in eV)

To the left of the element symbol the atomic weight and the element number are given		To the left of the element symbol the atomic weight and the element number are given	
1.008 H 1	13.595	26.98 Al 13	5.98; 18.8; 28.4
4.003 He 2	24.59; 54.38	28.09 Si 14	8.15; 16.3; 33.4
6.94 Li 3	5.39; 75.6; 122.4	30.98 P 15	10.48; 19.6; 30.2
9.013 Be 4	9.32; 18.2; 153.8	32.07 S 16	10.36; 23.4; 35.0
10.82 B 5	8.30; 25.2; 37.9	35.46 Cl 17	13.01; 23.8; 39.9
12.01 C 6	11.26; 24.4; 47.9	39.94 Ar 18	15.76; 27.7; 40.9
14.01 N 7	14.53; 29.6; 47.4	39.10 K 19	4.34; 31.8; 46.0
16.00 O 8	13.61; 35.2; 54.9	40.08 Ca 20	6.11; 11.9; 5.12
19.00 F 9	17.42; 35.0; 62.6	44.96 Sc 21	6.54; 12.8; 24.75
20.19 Ne 10	21.56; 41.1; 64.0	47.90 Ti 22	6.82; 13.6; 27.5
22.99 Na 11	5.14; 47.3; 71.6	50.95 V 23	6.74; 14.65; 29.3
24.32 Mg 12	7.64; 15.0; 80.1	52.01 Cr 24	6.76; 16.5; 30.95

Continued

To the left of the element symbol the atomic weight and the element number are given		To the left of the element symbol the atomic weight and the element number are given	
54.94 Mn 25	7.43; 15.6; 33.7	87.63 Sr 38	5.69; 10.38
55.85 Fe 26	7.90; 16.2; 30.6	88.92 Y 39	6.38; 12.2; 20.5
58.94 Co 27	7.86; 17.05; 33.5	91.22 Zr 40	6.84; 13.1; 23.0
58.66 Ni 28	7.63; 18.15; 35.2	92.91 Nb 41	6.88; 14.3; 25.0
63.54 Cu 29	7.72; 20.3; 36.8	95.95 Mo 42	7.1; 16.15; 27.1
65.38 Zn 30	9.39; 18.0; 39.7	99.0 Tc 43	7.28; 15.3
69.72 Ga 31	6.00; 20.4; 30.6	101.1 Ru 44	7.36; 16.8; 28.5
72.60 Ge 32	7.88; 15.9; 34.1	102.9 Rh 45	7.46; 18.1; 31.05
74.91 As 33	9.81; 20.1; 28.0	106.7 Pd 46	8.33; 19.4; 32.9
78.96 Se 34	9.75; 21.3; 33.9	107.9 Ag 47	7.57; 21.5; 34.8
79.92 Br 35	11.84; 19.1; 25.7	112.4 Cd 48	8.99; 16.9; 37.5
83.80 Kr 36	14.00; 26.4; 36.8	114.8 In 49	5.78; 18.8; 27.9
85.48 Rb 37	4.19; 23.0; 36.0	119.7 Sn 50	7.34; 14.5; 30.5

Continued

To the left of the element symbol the atomic weight and the element number are given	To the left of the element symbol the atomic weight and the element number are given
121.8 Sb 8.64; 18.0; 24.7 51	156.9 Gd 11.4 64
127.6 Te 9.01; 18.6; 30.5 52	158.9 Tb 6.74 65
126.9 I 10.45; 19.4 53	162.5 Dy 6.82 66
131.3 Xe 12.13; 21.1; 32.0 54	164.9 Ho 6.9 67
132.9 Cs 3.89; 23.4; 35.0 55	167.2 Er 6.9 68
137.4 Ba 5.21; 10.0; 56.0 56	168.9 Tu 6.9 69
138.9 La 5.61; 11.1; 20.4 57	173.0 Yb 6.2 70
140.1 Ce 6.91; 14.8; 15.8 58	175.0 Lu 5.0 71
140.9 Pr 5.76 59	178.6 Hf 7.0; 14.9 72
144.3 Nd 6.31 60	181.0 Ta 7.88; 16.2 73
145.0 Pm 6.3 61	183.9 W 7.98; 17.7 74
150.4 Sm 5.1; 11.4 62	186.3 Re 7.87; 16.6 75
152.0 Eu 5.67; 11.4 63	190.2 Os 8.7; 17.0 76

Continued

To the left of the element symbol the atomic weight and the element number are given		To the left of the element symbol the atomic weight and the element number are given	
192.2		210.0	
Ir	9.0	At	9.4
77		85	
195.2		222.0	
Pt	9.0; 18.6	Rn	10.75
78		86	
197.0		223.0	
Au	9.22; 10.5	Fr	4.0
79		87	
200.6		226.0	
Hg	10.44; 18.75; 34.2	Ra	5.28; 10.1
80		88	
204.4		227.0	
Tl	6.11; 20.3; 29.7	Ac	5.5
81		89	
207.2		232.0	
Pb	7.42; 15.0; 31.9	Th	5.7; 19.4
82		90	
209.0		231.0	
Bi	7.29; 16.6; 25.4	Pa	5.7
83		91	
210.0		238.9	
Po	8.3	U	4.0
84		92	

2) LOW ATOMIC TERMS

The table presents the low terms for the abundant elements [1]. The values are given in cm^{-1} relative to the ground state. The left column gives the fundamental term and the ionization potential in cm^{-1}.

Atom	Term	Term value	Atom	Term	Term value
H($^2S_{1/2}$)	$2^2P_{1/2}$	82258.907		3^3S_1	183236.9
109675.77	$2^2S_{1/2}$	82258.942		3^1S_0	184864.9
	$2^2P_{3/2}$	82259.272		$3^3P_{0,1,2,}$	185565
	$3^2P_{1/2}$	97492.2	Li($^1S_0$)	$2^2P_{1/2}$	14903.7
He($^1S_0$)	2^3S_1	159856.1	43487.19	$2^2P_{3/2}$	14904.0
198305.0	2^1S_0	163277.5		$3^2S_{1/2}$	27206.1
	$2^3P_{0,1,2,}$	169086.9		$3^2P_{1/2,3/2}$	30925.4
	2^1P_1	171135.0		$3^2D_{1/2,3/2}$	31283.1

Continued

Atom	Term	Term value	Atom	Term	Term value
	$4^2S_{1/2}$	35012.1			150774.07
					150860.47
C($^3P_0$)	$^3P_1$	16.4			
					151040.41
90878.3	$^3P_2$	43.4			
					152972.70
	$^1D_2$	10192.6			
	$^1S_0$	21648.4	Na($^2S_{1/2}$)	$^2P_{1/2}$	16956.18
	$^5S_2$	33735.2	41449.65	$^2P_{3/2}$	16973.38
	$^3P_0$	60333.8		$^2S_{1/2}$	25739.86
	$^3P_1$	60353.0		$^2D_{3/2}$	29172.855
	$^3P_2$	60393.5		$^2D_{5/2}$	29172.904
				$^2P_{1/2}$	30266.88
N($^4S_{3/2}$)	$^2D_{5/2}$	19224.5			
				$^2P_{3/2}$	30272.51
117345.0	$^2D_{3/2}$	19233.1			
	$^2P_{1/2,\,3/2}$	28838.9	Mg($^1S_0$)	$^3P_0$	21850.4
	$^4P_{1/2}$	83284.1	61669.14	$^3P_1$	21870.4
	$^4P_{3/2}$	83317.8		$^3P_2$	21911.1
	$^4P_{5/2}$	83364.6		$^1P_1$	35051.4
				$^3S_1$	41197.4
O($^3P_2$)	$^3P_1$	158.3			
				$^1S_0$	43503.0
109836.7	$^3P_0$	227.0			
	$^1D_2$	15867.7	Al($^2P_{1/2}$)	$^2P_{3/2}$	112.0
	$^1S_0$	33792.4	48279.16	$^2S_{1/2}$	25347.7
	$^5S_2$	73767.8		$^4P_{1/2}$	29020.3
	$^3S_1$	76794.7		$^4P_{3/2}$	29066.9
				$^4P_{5/2}$	29142.7
F($^2P_{3/2}$)	$^2P_{1/2}$	404.1			
140553.5	$^4P_{5/2}$	102406.5	S($^3P_2$)	$^3P_1$	396.8
	$^4P_{3/2}$	102681.2	83559.3	$^3P_0$	573.6
	$^4P_{1/2}$	102841.2		$^1D_2$	9239.0
	$^2P_{3/2}$	104731.9		$^1S_0$	22181.4
	$^2P_{1/2}$	105057.1		$^5S_2$	52623.9
				$^3S_1$	55331.1
Ne($^1S_0$)	$2p^5(^2P_{3/2})3s$	134043.79			
173931.7		134461.24	Cl($^2P_{3/2}$)	$^2P_{1/2}$	881.0
	$2p^5(^2P_{1/2})3s$	134820.59	104991.0	$^4P_{5/2}$	71954.0
		135890.67		$^4P_{3/2}$	72484.2
	$2p^5(^2P)3p$	148259.75		$^4P_{1/2}$	72822.6
		149659.00		$^2P_{3/2}$	74221.4
		149826.18		$^2P_{1/2}$	74861.2
		150123.55			
		150317.82	Ar($^1S_0$)	$3p^5(^2P_{3/2})4s$	93143.8
		150919.39	127109.9		93750.6

Continued

Atom	Term	Term value	Atom	Term	Term value
	$3p^5(^2P_{1/2})4s$	94553.7	Br($^2P_{3/2}$)	$^2P_{1/2}$	3685.0
		95399.9	95550.0	$^4P_{5/2}$	63429.8
	$3p^5(^2P)4p$	104102.1		$^4P_{3/2}$	64900.5
		105462.8		$^4P_{1/2}$	66877.2
		105615.3		$^2P_{3/2}$	67176.9
		106087.3		$^2P_{1/2}$	68963.5
		106237.6			
		107054.3	Kr($^1S_0$)	$4p^5(^2P_{3/2})5s$	79972.5
		107131.7	112915.2		80917.6
		107289.7		$4p^5(^2P_{1/2})5s$	85192.4
		107496.4			85847.5
		108722.7		$4p^5(^2P)5p$	91169.3
					92295.2
K($^1S_{1/2}$)	$^2P_{1/2}$	12985.2			92308.2
35009.78	$^2P_{3/2}$	13042.9			92965.2
	$^2S_{1/2}$	21026.8			93124.1
	$^2D_{3/2}$	21534.4			94093.7
	$^2D_{5/2}$	21536.7			97596.7
	$^2P_{1/2}$	24701.4			97919.9
	$^2P_{3/2}$	24720.2			97946.0
					98855.9
Ca($^1S_0$)	$^3P_0$	15157.9			
49304.8	$^3P_1$	15210.0	Rb($^2S_{1/2}$)	$^2P_{1/2}$	12578.96
	$^3P_2$	15316.0	33691.0	$^2P_{3/2}$	12816.6
	$^3D_1$	20335.3		$^2D_{5/2}$	19355.01
	$^3D_2$	20349.2		$^2D_{3/2}$	19355.45
	$^3D_3$	20370.9		$^2S_{1/2}$	20133.6
	$^1D_2$	21850.0		$^2P_{1/2}$	23715.2
Cu($^2S_{1/2}$)	$^2D_{5/2}$	11202.5		$^2P_{3/2}$	23792.7
62317.2	$^2D_{3/2}$	13245.4		$^2D_{3/2}$	25700.5
	$^2P_{1/2}$	30535.3		$^2D_{5/2}$	25703.5
	$^2P_{3/2}$	30783.7	Sr($^1S_0$)	$^3P_0$	14317.5
Zn($^1S_0$)	$^3P_0$	32311.3	45925.6	$^3P_1$	14504.4
75766.8	$^3P_1$	32501.4		$^3P_2$	14898.5
	$^3P_2$	32890.3		$^3D_0$	18159.0
				$^3D_1$	18218.8
Ga($^2P_{1/2}$)	$^2P_{3/2}$	826.2		$^3D_2$	18319.3
48380.0	$^2S_{1/2}$	24788.6		$^1D_2$	20149.7
	$^2P_{1/2}$	33044.1	In($^2P_{1/2}$)	$^2P_{3/2}$	2212.6
	$^2P_{3/2}$	33155.0	46669.9	$^2S_{1/2}$	24372.8

Continued

Atom	Term	Term value	Atom	Term	Term value
I($^2P_{3/2}$)	$^2P_{1/2}$	7603.2		$^2D_{3/2}$	22588.9
84340.0	$^4P_{5/2}$	54633.5		$^2D_{5/2}$	22631.8
	$^4P_{3/2}$	61819.8		$^2S_{1/2}$	24317.2
	$^4P_{1/2}$	60896.3		$^2F_{5/2}$	24472.3
	$^2P_{3/2}$	56092.9		$^2F_{7/2}$	24472.5
	$^2P_{1/2}$	63186.8	Ba($^1S_0$)	$^3D_1$	9034.0
Xe($^1S_0$)	$5p^5(^2P_{3/2})6s$	67068.0	42032.4	$^3D_2$	9215.5
97834.4		68045.7		$^3D_3$	9596.5
	$5p^5(^2P_{1/2})6s$	76197.3		$^1D_2$	11395.3
		77269.6		$^3P_0$	12266.0
	$5p^5(^2P_{3/2})6p$	77285.6		$^3P_1$	12636.6
		78120.3		$^3P_2$	13514.7
		78403.6	Hg($^1S_0$)	$^3P_0$	37645.0
		78956.5	84184.1	$^3P_1$	39412.3
		79213.0		$^3P_2$	44043.0
		80119.5		$^1P_0$	54068.8
	$5p^5(^2P_{1/2})6p$	88379.6		$^3S_0$	62350.5
		89162.9		$^1S_0$	63928.2
		89279.2		$5d^96s^2(^2D_{5/2})\times$ $\times 6p^2P_1$	68886.6
		89860.5			
Cs($^2S_{1/2}$)	$^2P_{1/2}$	11178.2		$^3P_0$	69516.7
31406.71	$^2P_{3/2}$	11732.4		$^3P_1$	69661.9
	$^2D_{3/2}$	14499.5		$^3P_2$	71207.5
	$^2D_{5/2}$	14597.0	Tl($^2P_{1/2}$)	$^2P_{3/2}$	7792.7
	$^2S_{1/2}$	18535.5	49264.2	$^2S_{1/2}$	26477.5
	$^2P_{1/2}$	21765.6		$^2P_{1/2}$	34159.9
	$^2P_{3/2}$	21946.7		$^2P_{3/2}$	35161.1

3) OSCILLATOR STRENGTHS FOR ATOMS [2]

Atom	Transition	Wavelength (Å)	Oscillator strength
H	1*S*-2*P*	1 215.7	0.416
	1*S*-3*P*	1 025.7	0.079
	1*S*-4*P*	972.5	0.029
	2*S*-3*P*	6 562.7	0.435
	2*P*-3*S*	6 562.5	0.014

Continued

Atom	Transition	Wavelength (Å)	Oscillator strength
	$2P$-$3D$	6 562.8	0.696
	$2S$-$4P$	4 861	0.103
	$2P$-$4S$	4 861	0.0031
	$2P$-$4D$	4 861	0.122
He	1^1S-2^1P	584.3	0.276
	1^1S-3^1P	537.0	0.0734
	1^1S-4^1P	522.2	0.030
	2^1S-2^1P	20 581.3	0.376
	2^1S-3^1P	5 015.7	0.151
	2^1S-4^1P	3 964.7	0.14
	2^3S-2^3P	10 830	0.539
	2^3S-3^3P	3 888.6	0.0645
	2^3S-4^3P	3 187.7	0.023
	2^1P-3^1S	7 281.3	0.049
	2^1P-4^1S	5 048	0.008
	2^1P-3^1D	6 678.1	0.711
	2^1P-4^1D	4 921.9	0.121
	2^3P-3^3S	7 065	0.076
	2^3P-4^3S	4 713	0.012
	2^3P-3^3D	5 876	0.623
	2^3P-4^3D	4 472	0.123
Li	$2^2S_{1/2}$-$2^2P_{1/2}$	6 707.7	0.248
	$2^2S_{1/2}$-$2^2P_{3/2}$	6 707.9	0.496
	$2^2P_{1/2}$-$3^2S_{1/2}$	8 126.5	0.109
	$2^2P_{3/2}$-$3^2S_{1/2}$	8 126.3	0.109
Na	$3^2S_{1/2}$-$3^2P_{1/2}$	5 889.9	0.324
	$3^2S_{1/2}$-$3^2P_{3/2}$	5 895.9	0.648
	$3^2S_{1/2}$-$4^2P_{1/2}$	3 303.9	0.051
	$3^2S_{1/2}$-$4^2P_{3/2}$	3 302.9	0.102
	$3^2P_{1/2}$-$4^2S_{1/2}$	22 084.0	0.167
	$3^2P_{3/2}$-$4^2S_{1/2}$	22 057.0	0.335
K	$4^2S_{1/2}$-$4^2P_{1/2}$	7 699.0	0.347
	$4^2S_{1/2}$-$4^2P_{3/2}$	7 664.9	0.684
	$4^2P_{1/2}$-$5^2S_{1/2}$	12 434.3	0.051
	$4^2P_{3/2}$-$5^2S_{1/2}$	12 523.0	0.102
Rb	$5^2S_{1/2}$-$5^2P_{1/2}$	7 947.6	0.363
	$5^2S_{1/2}$-$5^2P_{3/2}$	7 800.2	0.726

Continued

Atom	Transition	Wavelength (Å)	Oscillator strength
	$5^2P_{1/2}$-$6^2S_{1/2}$	13 237.3	0.187
	$5^2P_{3/2}$-$6^2S_{1/2}$	13 667.0	0.364
Cs	$6^2S_{1/2}$-$6^2P_{1/2}$	8 943.5	0.394
	$6^2S_{1/2}$-$6^2P_{3/2}$	8 521.2	0.814
	$6^2S_{1/2}$-$7^2P_{1/2}$	4 593.2	2.84×10^{-3}
	$6^2S_{1/2}$-$7^2P_{3/2}$	4 555.4	0.0017
	$6^2P_{1/2}$-$5^2D_{3/2}$	30 100	0.251
	$6^2P_{3/2}$-$5^2D_{3/2}$	36 127	0.0211
	$6^2P_{3/2}$-$5^2D_{5/2}$	34 892	0.204
	$6^2P_{3/2}$-$7^2S_{1/2}$	13 588	0.208
	$6^2P_{1/2}$-$7^2S_{1/2}$	14 695	0.171
	$6^2P_{1/2}$-$6^2D_{3/2}$	8 761.4	0.298
	$6^2P_{3/2}$-$6^2D_{3/2}$	9 208.5	0.0397
	$6^2P_{3/2}$-$6^2D_{5/2}$	9 172.3	0.332
	$5^2D_{5/2}$-$4F$	10 123.6	0.324
	$5^2D_{3/2}$-$4F$	10 024.4	0.302
Mg	3^1S_0-3^1P_1	2 852	1.2
	3^1S_0-3^3P_1	4 571.1	2.6×10^{-6}
Ca	4^1S_0-4^1P_1	4 226.7	1.49
	4^3P_2-4^3D_3	4 454.8	0.26
	4^3P_1-4^3D_2	4 435.0	0.27
	4^3P_0-4^3D_1	4 425.4	0.32
	4^3P_2-4^3D_2	4 455.9	0.047
	4^3P_1-4^3D_1	4 435.7	0.084
	4^3P_2-4^3D_1	4 456.6	0.003
	4^1S_0-4^3P_1	6 572.8	4.2×10^{-5}
	4^3P_2-5^3S_1	6 162.2	0.10
	4^3P_1-5^3S_1	6 122.2	0.12
	4^3P_0-5^3S_1	5 102.7	0.12
Sr	5^1S_0-5^1P_1	4 607.3	1.54
	5^1S_0-5^3P_1	6 892.6	8.6×10^{-4}
	5^3P_2-6^3S_1	7 070.1	0.17
	5^3P_1-6^3S_1	6 878.4	0.17
	5^3P_0-6^3S_1	6 791.0	0.18
	5^3P_2-5^3D_3	4 962.2	0.33
	5^3P_1-5^3D_2	4 872.5	0.31
	5^3P_2-5^3D_2	4 967.9	0.06

Continued

Atom	Transition	Wavelength (Å)	Oscillator strength
Ba	6^1S_0-6^1P_1	5 535.5	1.40
	6^1S_0-6^3P_1	7 911.3	0.085
	6^3P_2-7^3S_1	7 906.0	0.31
	6^3P_1-7^3S_1	7 392.0	0.36
	6^3P_0-7^3S_1	7 195.0	0.50
	6^3P_2-6^3D_3	5 778.0	0.91
	6^3P_2-6^3D_2	5 800.0	0.21
Zn	4^1S_0-4^1P_1	2 139.0	1.2
	4^1S_0-4^3P_1	3 076.0	1.7×10^{-4}
Cd	5^1S_0-5^1P_1	2 288.0	1.2
	5^1S_0-5^3P_1	3 261.0	0.0018
Hg	6^1S_0-6^1P_1	1 849.0	1.2
	6^1S_0-6^3P_1	2 537.0	0.03
	6^3P_2-7^3S_1	5 461.0	0.16
	6^3P_1-7^3S_1	4 358.0	0.15
	6^3P_0-7^3S_1	4 047.0	0.21
	6^3P_1-7^1S_0	4 078.0	0.006
	6^3P_2-6^3D_3	3 650.0	0.3
	6^3P_1-6^3D_1	3 131.6	0.2
	6^3P_1-6^3D_2	3 126.0	0.1
	6^3P_0-6^3D_1	2 967.0	0.4
	6^3P_1-6^1D_2	3 131.8	0.04
Al	$3^2P_{1/2}$-$4^2S_{1/2}$	3 944.0	0.15
	$3^2P_{3/2}$-$4^2S_{1/2}$	3 962.0	0.15
	$3^2P_{3/2}$-$3^2D_{3/2,\,5/2}$	3 092.0	0.23
	$3^2P_{1/2}$-$3^2D_{3/2}$	3 089.0	0.22
	$3^2P_{1/2}$-$5^2S_{1/2}$	3 652.0	0.023
	$3^2P_{3/2}$-$5^2S_{1/2}$	2 660.0	0.024
	$3^2P_{3/2}$-$4^2D_{3/2,\,5/2}$	2 575.0	0.07
	$3^2P_{1/2}$-$4^2D_{3/2}$	2 568.0	0.07
Ga	$4^2P_{1/2}$-$5^2S_{1/2}$	4 033.0	0.129
	$4^2P_{3/2}$-$5^2S_{1/2}$	4 172.0	0.135
	$4^2P_{3/2}$-$4^2D_{5/2}$	2 943.6	0.29
	$4^2P_{3/2}$-$4^2D_{3/2}$	2 944.2	0.038
	$4^2P_{1/2}$-$4^2D_{3/2}$	2 874.0	0.32
	$4^2P_{3/2}$-$6^2S_{1/2}$	2 719.7	0.014
	$4^2P_{1/2}$-$6^2S_{1/2}$	2 659.9	0.014

Continued

Atom	Transition	Wavelength (Å)	Oscillator strength
	$4^2P_{1/2}$-$5^2D_{3/2}$	2 450.1	0.037
	$4^2P_{3/2}$-$5^2D_{5/2}$	2 500.2	0.037
In	$5^2P_{1/2}$-$6^2S_{1/2}$	4 101.8	0.144
	$5^2P_{3/2}$-$6^2S_{1/2}$	4 511.3	0.157
	$5^2P_{1/2}$-$5^2D_{3/2}$	3 039.4	0.36
	$5^2P_{3/2}$-$5^2D_{5/2}$	3 256.1	0.37
	$5^2P_{3/2}$-$5^2D_{3/2}$	3 258.6	0.06
	$5^2P_{1/2}$-$7^2S_{1/2}$	2 753.9	0.017
	$5^2P_{3/2}$-$7^2S_{1/2}$	2 932.6	0.017
	$5^2P_{1/2}$-$6^2D_{3/2}$	2 560.2	0.045
	$5^2P_{3/2}$-$6^2D_{5/2}$	2 710.3	0.052
	$5^2P_{3/2}$-$6^2D_{3/2}$	2 713.4	0.006
Tl	$6^2P_{1/2}$-$7^2S_{1/2}$	3 776.0	0.133
	$6^2P_{3/2}$-$7^2S_{1/2}$	5 350.0	0.151
	$6^2P_{3/2}$-$6^2D_{5/2}$	3 519.0	0.346
	$6^2P_{3/2}$-$6^2D_{3/2}$	3 529.0	0.04
	$6^2P_{1/2}$-$6^2D_{3/2}$	2 768.0	0.29
Ne	2^1S_0-2^3P_1	743.72	0.012
	2^1S_0-2^1P_1	735.895	0.12
Ar	3^1S_0-3^3P_1	1 066.6	0.036
	3^1S_0-3^1P_1	1 048.2	0.28
Kr	4^1S_0-4^3P_1	1 235.8	0.16
	4^1S_0-4^1P_1	1 164.6	0.14
Xe	5^1S_0-5^3P_1	1 469.6	0.26
	5^1S_0-5^1P_1	1 295.6	0.27

4) ELECTRON BINDING ENERGY E IN THE NEGATIVE ION [3]

Atomic No.	Ion	E (eV)	Atomic No.	Ion	E (eV)
1	$H^-(^1S)$	0.75421	6	$C^-(^4S)$	1.263
2	$He^-(^4P)$	0.076		$C^-(^2D)$	0.034
3	$Li^-(^1S)$	0.617	7	N^-	—
4	$Be^-(^2P)$	—	8	$O^-(^2P)$	1.463
5	$B^-(^3P)$	0.3	9	$F^-(^1S)$	3.400

Continued

Atomic No.	Ion	E(eV)	Atomic No.	Ion	E(eV)
10	Ne^-	—	40	$\mathrm{Zr}^-(^4F)$	0.8
11	$\mathrm{Na}^-(^1S)$	0.546	41	$\mathrm{Nb}^-(^5D)$	1.1
12	Mg^-	—	42	$\mathrm{Mo}^-(^6S)$	1.1
13	$\mathrm{Al}^-(^3P)$	0.5	43	$\mathrm{Tc}^-(^5\mathrm{D})$	1.0
	$\mathrm{Al}^-(^1D)$	0.095	44	$\mathrm{Ru}^-(^4F)$	1.5
14	$\mathrm{Si}^-(^4S)$	1.388	45	$\mathrm{Rh}^-(^3F)$	1.5
	$\mathrm{Si}^-(^2P)$	0.524	46	$\mathrm{Pd}^-(^2D)$	1.2
	$\mathrm{Si}^-(^2P)$	0.034	47	$\mathrm{Ag}^-(^1S)$	1.30
15	$\mathrm{P}^-(^3P)$	0.746	48	$\mathrm{Cd}^-(^2P)$	—
16	$\mathrm{S}^-(^2P)$	2.077	49	$\mathrm{In}^-(^3P)$	0.5
17	$\mathrm{Cl}^-(^1S)$	3.616	50	$\mathrm{Sn}^-(^4S)$	1.19
18	Ar^-	—	51	$\mathrm{Sb}^-(^3P)$	1.15
19	$\mathrm{K}^-(^1S)$	0.5015	52	$\mathrm{Te}^-(^2P)$	1.9
20	Ca^-	—	53	$\mathrm{I}^-(^1S)$	3.066
21	$\mathrm{Sc}^-(^3F)$	—	54	Xe^-	—
22	$\mathrm{Ti}^-(^4F)$	0.2	55	$\mathrm{Cs}^-(^1S)$	0.471
23	$\mathrm{V}^-(^5D)$	0.7	56	$\mathrm{Ba}^-(^2D)$	—
24	$\mathrm{Cr}^-(^6S)$	0.66	57	$\mathrm{La}^-(^3F)$	0.5
25	$\mathrm{Mn}^-(^5D)$	—	72	$\mathrm{Hf}^-(^4F)$	—
26	$\mathrm{Fe}^-(^4F)$	0.4	73	$\mathrm{Ta}^-(^5D)$	0.8
27	$\mathrm{Co}^-(^3F)$	0.9	74	$\mathrm{W}^-(^6S)$	0.5
28	$\mathrm{Ni}^-(^2D)$	1.2	75	$\mathrm{Re}^-(^5D)$	0.15
29	$\mathrm{Cu}^-(^1S)$	1.22	76	$\mathrm{Os}^-(^4F)$	1.4
30	Zn^-	—	77	$\mathrm{Ir}^-(^3F)$	2.0
31	$\mathrm{Ga}^-(^3P)$	0.5	78	$\mathrm{Pt}^-(^2D)$	2.13
32	$\mathrm{Ge}^-(^4S)$	1.15	79	$\mathrm{Au}^-(^1S)$	2.309
33	$\mathrm{As}^-(^3P)$	0.80	80	$\mathrm{Hg}^-(^2P)$	—
34	$\mathrm{Se}^-(^2P)$	2.021	81	$\mathrm{Tl}^-(^3P)$	0.5
35	$\mathrm{Br}^-(^1S)$	3.365	82	$\mathrm{Pb}^-(^4S)$	1.2
36	Kr^-	—	83	$\mathrm{Bi}^-(^3P)$	1.0
			84	$\mathrm{Po}^-(^2P)$	1.6
37	$\mathrm{Rb}^-(^1S)$	0.486	85	$\mathrm{At}^-(^1S)$	2.9
38	$\mathrm{Sr}^-(^2\mathrm{D})$	—	86	Rn^-	—
39	$\mathrm{Y}^-(^3F)$	—	87	$\mathrm{Fr}^-(^1S)$	0.5

5) POLARIZABILITY a_0^3 OF ATOMS [4]
(a_0 IS THE BOHR RADIUS)

Atom	a_0^3	Atom	a_0^3	Atom	a_0^3
H	4.50	F	3.8	Ca	170
He(1^1S)	1.383	Ne	2.68	Br	26
He(2^3S)	320	Na	162	Kr	16.74
He(2^1S)	800	Mg	12	Rb	310
Li	162	P	28	Sr	200
Be	38	S	25	Xe	27.06
B	20	Cl	17	Cs	385
C	12	Ar	11.08	Ba	280
N	7.5	K	287	Hg	34
O	5.4				

6) MAGNETIC SUSCEPTIBILITY χ OF ATOMS [4]

Atom	(10^{-6} cm^3 · mole^{-1})	Atom	(10^{-6} cm^3 · mole^{-1})	Atom	(10^{-6} cm^3 · mole^{-1})
He	1.88	Mg	23	Kr	29
Be	13	Ar	19	Xe	45
Ne	7.0	Ca	44		

7) POLARIZABILITY a_0^3 OF MOLECULES [4]
(a_0 IS THE BOHR RADIUS)

Molecule	a_0^3	Molecule	a_0^3	Molecule	a_0^3
Br_2	44	HBr	24	NO	11.5
CCl_4	69	HCl	17	NO_2	20
CF_4	19	HF	5.6	N_2	11.74
CH_4	17.3	HJ	35	N_2O	20
CO	13.2	H_2	5.178	O_2	10.6
CO_2	17.5	H_2O	9.8	SF_6	30
Cl_2	31	H_2S	25	SO_2	26
D_2	5.39	NH_3	14.8		

8) IONIZATION POTENTIALS I OF MOLECULES [5]

Molecule	I (eV)	Molecule	I (eV)	Molecule	I (eV)
Br_2	10.6	F_2	15.8	KI	8.3
BrCl	11.1	H_2	15.42	LiI	8.5
C_2	12.0	HBr	11.7	N_2	15.6
CH	11.1	HCl	12.8	NH	13.1
CH_2	11.8	HF	15.8	NH_3	10.3
CH_3	9.86	HI	10.4	NO	9.25
CH_4	13.0	HO_2	11.5	N_2O	12.7
CN	15.1	H_2O_2	10.9	NO_2	9.8
CO	14.01	H_2O	12.6	NaI	8.8
CO_2	13.8	H_2S	10.45	O_2	12.2
Cl_2	11.5	I_2	9.35	OH	13.2
ClO	10.4	IBr	10.3	O_3	11.7
ClO_2	11.1	ICl	10.4	SO_2	12.3

9) PARAMETERS OF DIATOMIC MOLECULES IN THE GROUND ELECTRON STATE [4]

The energies of the vibrational E_v and rotational E_J states are expressed in terms of these parameters:

$$E_v = \hbar\omega_e\left(v + \frac{1}{2}\right) - \hbar\omega_e x_e\left(v + \frac{1}{2}\right)^2$$

$$E_J = \left[B_e - \alpha_e\left(v + \frac{1}{2}\right)\right] J(J + 1)$$

Here v is the vibrational quantum number and J is the rotational quantum number.

Notation in the table: μ is the reduced mass, D is the equilibrium distance between the nuclei

Molecule	Electron state	μ (amu)	$\hbar\omega_e$ (cm^{-1})	$\hbar\omega_e x_e$ (cm^{-1})	B_e (cm^{-1})	α_e (0.001 cm^{-1})	D (Å)	Dissociation energy (eV)
Ag_2	$X^1\Sigma_g^+$	53.93	207.0	0.643	0.496	0.19	2.5	1.68
AgCl	$X^1\Sigma^+$	26.68	34.34	1.15	0.123	0.6	2.281	3.24
AgH	$X^1\Sigma^+$	0.999	1 759	34.1	6.449	200	1.618	2.40
Al_2	$X^3\Sigma_g^-$	13.49	350	2.02	0.205	1.2	2.560	2.05
AlCl	$X^1\Sigma^+$	15.32	481.3	1.95	0.248	1.6	2.130	5.10
AlF	$X^1\Sigma^+$	11.15	802.8	4.86	0.552	4.95	1.654	6.85
AlH	$X^1\Sigma^+$	0.972	1 682.6	29.1	6.39	0.18	1.648	2.90
AlS	$X^2\Sigma^+$	10.04	979.2	6.9	0.641	5.8	1.618	5.14
Ar_2	$X^1\Sigma_g^+$	19.97	30.68	2.56	0.060	4.0	3.8	$9.5 \cdot 10^{-3}$

Continued

Molecule	Electron state	μ (amu)	$\hbar\omega_e$ (cm^{-1})	$\hbar\omega_e x_e$ (cm^{-1})	B_e (cm^{-1})	α_e (0.001 cm^{-1})	D (Å)	Dissociation energy (eV)
As_2	$X^1\Sigma_g^+$	37.46	429.6	1.12	0.102	0.33	2.104	2.96
Au_2	$X^1\Sigma_g^+$	98.48	190.9	0.42	0.028	0.072	2.472	2.34
B_2	$X^3\Sigma_g^-$	5.41	1 051	9.4	1.212	14	1.589	2.90
BBr	$X^1\Sigma^+$	9.52	684.3	3.52	0.490	3.5	1.89	4.30
BCl	$X^1\Sigma^+$	8.255	839.1	5.11	0.684	6.4	1.716	5.10
BF	$X^1\Sigma^+$	6.89	1 402	11.8	1.517	0.02	1.263	7.85
BH	$X^1\Sigma^+$	0.922	2 367	42.4	12.02	410	1.236	3.39
BO	$X^2\Sigma^+$	6.45	1 886	11.8	1.782	17	1.204	8.28
BaH	$X^2\Sigma^+$	1.001	1 168	14.5	3.39	66	2.23	1.80
BaO	$X^1\Sigma^+$	7.933	669.8	2.05	0.312	1.2	1.94	5.70
BaS	$X^1\Sigma^+$	10.86	379.4	0.88	0.103	0.32	2.507	4.10
BeH	$X^2\Sigma^+$	0.907	2 058	35.5	10.31	300	1.297	2.31
BeH^+	$X^1\Sigma^+$	0.907	2 222	40	10.8	290	1.267	3.0
BeO	$X^1\Sigma^+$	5.765	1 487	11.8	1.651	19	1.331	4.60
BeS	$X^1\Sigma^+$	7.035	998	6.14	0.791	6.6	1.742	3.80
Bi_2	$X^1\Sigma_g^+$	104.5	172.7	0.34	0.0228	0.050	2.659	1.86
Br_2	$X^1\Sigma_g^+$	39.95	323.1	1.17	0.0811	0.32	2.281	1.97
BrCl	$X^1\Sigma^+$	24.56	440	1.6	0.150	—	2.138	2.23
BrO	$X^1\Sigma^+$	15.35	673	4.5	0.357	5.2	1.756	2.38
C_2	$X^1\Sigma_g^+$	6.006	1 855	13.3	1.820	17.6	1.242	6.25
C_2^-	$X^2\Sigma_g^+$	6.006	1 781	11.6	1.747	16.7	1.263	8.5
CH	$X^2\Pi$	0.93	2 859	63.3	14.19	0.53	1.120	3.45
CH^+	$X^1\Sigma^+$	0.93	2 740	73	14.2	4.1	1.131	3.8
CN	$X^2\Sigma^+$	6.47	2 069	13.1	1.90	17	1.172	7.75
CO	$X^1\Sigma^+$	6.86	2 169.8	13.3	1.931	17.5	1.128	11.09
CO^+	$X^2\Sigma^+$	6.86	2 214	15.2	1.977	19	1.115	8.34
CS	$X^1\Sigma^+$	8.74	1 285	8.5	0.820	5.9	1.535	7.35
Ca_2	$X^1\Sigma_g^+$	20.04	65.0	1.1	0.046	—	4.28	0.12
CaCl	$X^2\Sigma^+$	18.81	369.3	1.4	0.152	0.78	2.439	4.55
Cl_2	$X^1\Sigma_g^+$	17.73	559.7	2.70	0.244	1.53	1.988	2.48
ClF	$X^1\Sigma^+$	12.37	786.3	6.23	0.516	43.6	1.628	2.56
Cs_2	$X^1\Sigma_g^+$	66.45	42.03	0.083	—	—	—	0.45
CsCl	$X^1\Sigma^+$	27.99	2 142	0.74	0.072	0.34	2.906	4.58
Cu_2	$X^1\Sigma_g^+$	31.77	268.9	1.6	0.109	0.56	2.22	2.05
CuCl	$X^1\Sigma^+$	22.76	415.2	1.6	0.176	1.0	2.05	3.60
CuH	$X^1\Sigma^+$	0.992	1 940	37	7.94	250	1.463	2.72
F_2	$X^1\Sigma_g^+$	9.50	892	—	0.883	—	1.417	1.38
GaCl	$X^1\Sigma^+$	23.50	365	1.1	0.150	0.72	0.202	4.94
H_2	$X^1\Sigma_g^+$	0.504	4 401.2	121.3	60.85	3 062	0.741	4.478

Continued

Molecule	Electron state	μ (amu)	$\hbar\omega_e$ (cm^{-1})	$\hbar\omega_e x_e$ (cm^{-1})	B_e (cm^{-1})	α_e (0.001 cm^{-1})	D(Å)	Dissociation energy (eV)
H_2^+	$X^2\Sigma_g^+$	0.504	2 297	62	29.8	1 400	1.06	2.65
HBr	$X^1\Sigma^+$	0.995	2 649	45.2	8.35	230	1.414	3.75
HCl	$X^1\Sigma^+$	0.980	2 991	52.8	10.44	310	1.275	4.43
HCl^+	$X^2\Sigma^+$	0.980	2 675	53.5	9.95	320	1.315	4.69
HF	$X^1\Sigma^+$	0.975	4 139	90	21.0	800	0.917	5.86
HJ	$X^1\Sigma^+$	1.00	2 308	39.0	6.427	170	1.609	3.06
HS	$X^2\Pi$	0.977	2 690	45.5	9.60	280	1.345	3.53
He_2^+	$X^2\Sigma_u^+$	2.001	1 698	35	7.21	220	1.080	2.47
J_2	$X^1\Sigma_g^+$	63.45	214.5	0.62	0.0374	11	2.666	1.54
JCl	$X^1\Sigma^+$	27.71	384.3	1.5	0.114	0.54	2.321	2.15
K_2	$X^1\Sigma_g^+$	19.55	92.64	0.35	0.0562	0.22	3.92	0.51
KBr	$X^1\Sigma^+$	26.25	219.2	0.76	0.081	0.4	2.821	3.96
KCl	$X^1\Sigma^+$	18.59	279.8	1.2	0.79	0.9	2.667	4.37
Li_2	$X^1\Sigma_g^+$	3.47	351.3	2.6	0.673	7.2	2.67	1.02
LiCl	$X^1\Sigma^+$	5.80	643.3	4.5	0.706	8.0	2.021	4.91
LiH	$X^1\Sigma^+$	0.88	1 406	23	7.51	210	1.595	2.47
LiJ	$X^1\Sigma^+$	6.58	498.2	3.4	0.443	4.1	2.392	1.92
Mg_2	$X^1\Sigma_g^+$	12.15	51.1	1.64	0.0929	3.8	2.89	0.050
MgCl	$X^1\Sigma^+$	14.42	491	2.0	0.245	1.6	2.198	3.24
N_2	$X^1\Sigma_g^+$	7.003	2 358	14.1	1.998	17.8	1.098	9.76
N_2^+	$X^2\Sigma_g^+$	7.003	2 207	16	1.932	20	1.116	8.73
NCl	$X^3\Sigma^-$	10.04	827	5.1	0.647	—	1.608	2.82
NH	$X^3\Sigma^-$	0.94	3 203	87	16.7	650	1.045	3.20
NO	$X^2\Pi_{1/2}$	7.47	1 904	14	1.705	18	1.151	6.50
NO^+	$X^1\Sigma^+$	7.47	2 377	16	2.002	20	1.062	10.9
NS	$X^2\Pi$	9.75	1 219	7.5	0.774	6.1	1.495	4.80
Na_2^+	$X^2\Sigma_g^+$	11.50	159.1	0.73	0.155	0.86	2.077	1.0
NaCl	$X^1\Sigma^+$	13.95	364.6	1.78	0.218	1.6	2.361	4.27
NaF	$X^1\Sigma^+$	10.40	536	—	0.437	1.8	1.926	4.94
O_2	$X^3\Sigma_g^-$	8.00	1 580	12.0	1.446	16	1.207	5.12
O_2^+	$X^2\Pi_g$	8.00	1 905	16.3	1.689	0.20	1.117	6.66
O_2^-	$X^2\Pi_g$	8.00	1 098	12	1.17	—	1.341	4.1
OH	$X^2\Pi$	0.948	3 735	83	18.87	710	0.971	4.40
OH^+	$X^3\Sigma^-$	0.948	3 124	85	16.78	270	1.029	5.06
P_2	$X^1\Sigma_g^+$	15.49	781	2.8	0.304	1.4	1.894	5.04
PF	$X^3\Sigma^-$	11.78	846.8	4.5	0.566	4.6	1.590	4.50
PN	$X^1\Sigma^+$	9.645	1 337	7.0	0.786	5.6	1.187	7.57
PO	$X^2\Pi$	10.55	1 233	6.6	0.734	5.5	1.473	6.14
PbCl	$X^2\Pi_{1/2}$	30.27	304.2	0.89	0.119	—	2.179	3.77

Continued

Molecule	Electron state	μ (amu)	$\hbar\omega_e$ (cm^{-1})	$\hbar\omega_e x_e$ (cm^{-1})	B_e (cm^{-1})	α_e (0.001 cm^{-1})	D(Å)	Dissociation energy (eV)
PbO	$X^1\Sigma^+$	14.85	721.3	3.5	0.308	1.9	1.922	3.87
PtO	$X^1\Sigma^+$	14.79	851.1	5.0	0.382	2.8	1.727	2.84
RbBr	$X^1\Sigma^+$	41.30	169.4	0.46	0.0475	0.19	2.945	2.94
RbCl	$X^1\Sigma^+$	25.06	233.3	0.86	0.0876	0.45	2.787	4.46
S_2	$X^3\Sigma_g^-$	15.99	725.7	2.8	0.295	1.6	1.889	4.38
SO	$X^3\Sigma^-$	10.67	1 149	5.6	0.721	5.7	1.492	3.26
Si_2	$X^3\Sigma_g^-$	14.04	511.0	2.0	0.239	1.3	2.246	3.33
SiCl	$X^2\Pi$	15.67	535.9	2.2	0.256	1.6	2.058	4.50
SiH	$X^2\Pi$	0.973	2 042	36	7.50	220	1.520	3.06
SiH^+	$X^1\Sigma^+$	0.973	2 157	34	7.66	210	1.499	3.20
SiO	$X^1\Sigma^+$	10.19	1 241	5.9	0.727	5.0	1.510	8.20
SnO	$X^1\Sigma^+$	14.10	822.4	3.7	0.356	2.1	1.832	5.40
SrO	$X^1\Sigma^+$	13.53	653.3	4.0	0.338	2.1	1.920	5.70
TlCl	$X^1\Sigma^+$	30.21	287.5	1.2	0.0914	0.40	2.485	3.82
Xe_2	$X^1\Sigma_g^+$	65.65	21.3	0.7	0.013	0.4	4.45	0.024
ZnH	$X^2\Sigma^+$	0.993	1 608	55	6.68	250	1.594	0.85
ZnH^+	$X^1\Sigma^+$	0.993	1 916	39	7.41	250	1.51	2.50
ZrO	$X^1\Sigma^+$	13.61	969.8	5.0	0.423	2.3	1.711	7.80

10) DISSOCIATION ENERGIES E OF MOLECULAR IONS

Ion	E (eV)	Ion	E (eV)	Ion	E (eV)	Ion	E (eV)
Ar_2^+	1.2	Cs_2^+	0.7	I_2^+	2.8	Na_2^+	1.0
ArH^+	2.3	F_2^+	3.3	IBr^+	2.4	Ne_2^+	1.1
Br_2^+	3.3	H_2^+	2.65	ICl^+	2.5	NeH^+	2.2
C_2^+	5.5	HBr^+	4.0	K_2^+	0.75	O_2^+	6.66
CH^+	3.8	HCl^+	4.7	Kr_2^+	1.1	O_2^-	4.06
CO^+	8.35	HF^+	3.44	KrH^+	3.7	OH^+	5.06
CN^+	4.97	HI^+	3.1	Li_2^+	1.4	OH^-	4.77
CS^+	6.69	He_2^+	2.47	N_2^+	8.71	Rb_2^+	0.7
Cl_2^+	4.03	HeH^+	1.78	NO^+	10.9	Xe_2^+	1.0
ClO^+	4.7	HgH^+	2.3				

11) RUPTURE ENERGIES *E* OF MOLECULAR BONDS [5]

Reaction	*E* (eV)	Reaction	*E* (eV)
$CO_2 \rightarrow CO + O$	5.45	$(KI)_2 \rightarrow 2KI$	1.7
$CO_2 \rightarrow C + O_2$	11.4	$(LiBr)_2 \rightarrow 2LiBr$	2.0
$CH_4 \rightarrow CH_3 + H$	4.51	$(LiCl)_2 \rightarrow 2LiCl$	2.1
$HCO \rightarrow H + CO$	0.75	$(LiF)_2 \rightarrow 2LiF$	2.6
$HCN \rightarrow H + CN$	4.8	$(LiI)_2 \rightarrow 2LiI$	1.8
$CH_2O \rightarrow CHO + H$	3.77	$Li_2O \rightarrow LiO + Li$	3.8
$CH_2O_2 \rightarrow CHO + OH$	3.9	$LiOH \rightarrow Li + OH$	4.5
$C_2H_4 \rightarrow 2CH_2$	5.2	$LiOH \rightarrow LiO + H$	5.5
$C_2H_2 \rightarrow 2CH$	10.0	$NH_2 \rightarrow NH + H$	4.0
$CH_3CN \rightarrow CH_3 + CN$	4.5	$NH_3 \rightarrow NH_2 + H$	4.48
$CH_3Br \rightarrow CH_3 + Br$	2.9	$NO_2 \rightarrow NO + O$	3.11
$CH_3Cl \rightarrow CH_3 + Cl$	3.6	$N_2O \rightarrow N_2 + O$	1.67
$CH_3F \rightarrow CH_3 + F$	5.1	$N_2O \rightarrow NO + N$	4.93
$CH_3J \rightarrow CH_3 + J$	2.3	$NOH \rightarrow NO + H$	2.11
$CH_3OH \rightarrow CH_3 + OH$	3.9	$NOH \rightarrow N + OH$	4.2
$CH_3NH_2 \rightarrow CH_3 + NH_2$	3.5	$NOOH \rightarrow NO + OH$	2.1
$(CsBr)_2 \rightarrow 2CsBr$	1.5	$NOOH \rightarrow NO_2 + H$	3.3
$(CsCl)_2 \rightarrow 2CsCl$	1.6	$NF_2 \rightarrow NF + F$	3.4
$(CsF)_2 \rightarrow 2CsF$	1.7	$NaOH \rightarrow Na + OH$	3.9
$(CsI)_2 \rightarrow 2CsI$	1.5	$NaOH \rightarrow NaO + H$	4.8
$HO_2 \rightarrow H + O_2$	2.0	$(NaBr)_2 \rightarrow 2NaBr$	2.0
$HO_2 \rightarrow O + OH$	2.8	$(NaCl)_2 \rightarrow 2NaCl$	2.2
$H_2O \rightarrow H + OH$	5.12	$(NaF)_2 \rightarrow 2NaF$	2.6
$H_2O \rightarrow H_2 + O$	5.0	$(NaI)_2 \rightarrow 2NaI$	1.9
$H_2O_2 \rightarrow 2OH$	2.22	$O_3 \rightarrow O_2 + O$	1.04
$H_2O_2 \rightarrow HO_2 + H$	3.8	$(RbCl)_2 \rightarrow 2RbCl$	1.9
$H_2S \rightarrow HS + H$	4.0	$(RbF)_2 \rightarrow 2RbF$	2.1
$(KBr)_2 \rightarrow 2KBr$	2.0	$SO_2 \rightarrow SO + O$	5.66
$(KCl)_2 \rightarrow 2KCl$	1.8	$SO_2 \rightarrow S + O_2$	5.8
$(KF)_2 \rightarrow 2KF$	2.2	$SO_3 \rightarrow SO_2 + O$	3.5

12) DIPOLE MOMENT *d* (IN DEBYE UNITS = 10^{-18} STATCOULOMB-CM) OF SOME MOLECULES [6]

Molecule	Formula	*d*	Molecule	Formula	*d*
Hydrogen	H_2	0	Methyl bromide	CH_3Br	1.8
Nitrogen	N_2	0	Methyl chloride	CH_3Cl	1.86
Oxygen	O_2	0	Methyl iodide	CH_3I	1.6
Nitrogen oxide	NO	0.16	Methyl alcohol	CH_4O	1.65
Carbon monoxide	CO	0.1	Dichloromethane	CH_2Cl_2	1.6
Hydrogen bromide	HBr	0.8	Difluoromethane	CH_2F_2	1.96
Hydrogen chloride	HCl	1.1	Nitromethane	CH_3NO_2	3.1
Hydrogen fluoride	HF	1.9	Bromoform	$CHBr_3$	1.0
Lithium hydride	LiH	6.0	Chloroform	$CHCl_3$	1.2
Lithium bromide	LiBr	6.2	Trifluoromethane	CHF_3	1.6
Lithium fluoride	LiF	6.2	Triiodomethane	CHI_3	1.0
Lithium iodide	LiI	6.6	Acetyl chloride	C_2H_3OCl	2.5
Potassium bromide	KBr	9.0	Nitroethylene	$C_2H_3NO_2$	3.4
Potassium chloride	KCl	9.0	Acetonitrile	C_2H_3N	3.4
Cesium chloride	CsCl	10.4	Vinyl bromide	C_2H_3Br	1.41
Cesium fluoride	CsF	7.6	Vinyl chloride	C_2H_3Cl	1.44
Cesium iodide	CsI	11.0	Vinyl iodide	C_2H_3I	1.26
Carbon dioxide	CO_2	0	Ethylene oxide	C_2H_4O	1.9
Sulphur dioxide	SO_2	1.6	Acetic hydride	$(CH_3CO)_2O$	2.7
Hydrogen sulphide	H_2S	1.1	Acetic acid	$C_2H_4O_2$	1.7
Water	H_2O	1.84	Nitrobenzene	$C_6H_5NO_2$	3.8
Nitrogen dioxide	NO_2	0.3	Bromobenzene	C_6H_5Br	1.15
Nitrogen dioxide	N_2O	0.2	Chlorobenzene	C_6H_5Cl	1.6
Hydrogen cyanide	HCN	2.9	Benzonitrile	C_7H_5N	3.9
Bromide cyanide	CNBr	2.9	Dimethylaniline	C_8H_4N	1.6
Chlorine cyanide	CNCl	2.8	Anisole	C_7H_8N	1.2
Iodine cyanide	CNI	3.7	Acetone	$(CH_3)_2CO$	3.0
Ammonia	NH_3	1.4	Pyridine	C_5H_5N	2.2
Formaldehyde	CH_2O	2.2	Methyl ether	C_2H_6O	1.3
			Ethyl ether	$(C_2H_5)_2O$	1.1

13) POTENTIAL CURVES OF DIATOMIC MOLECULES

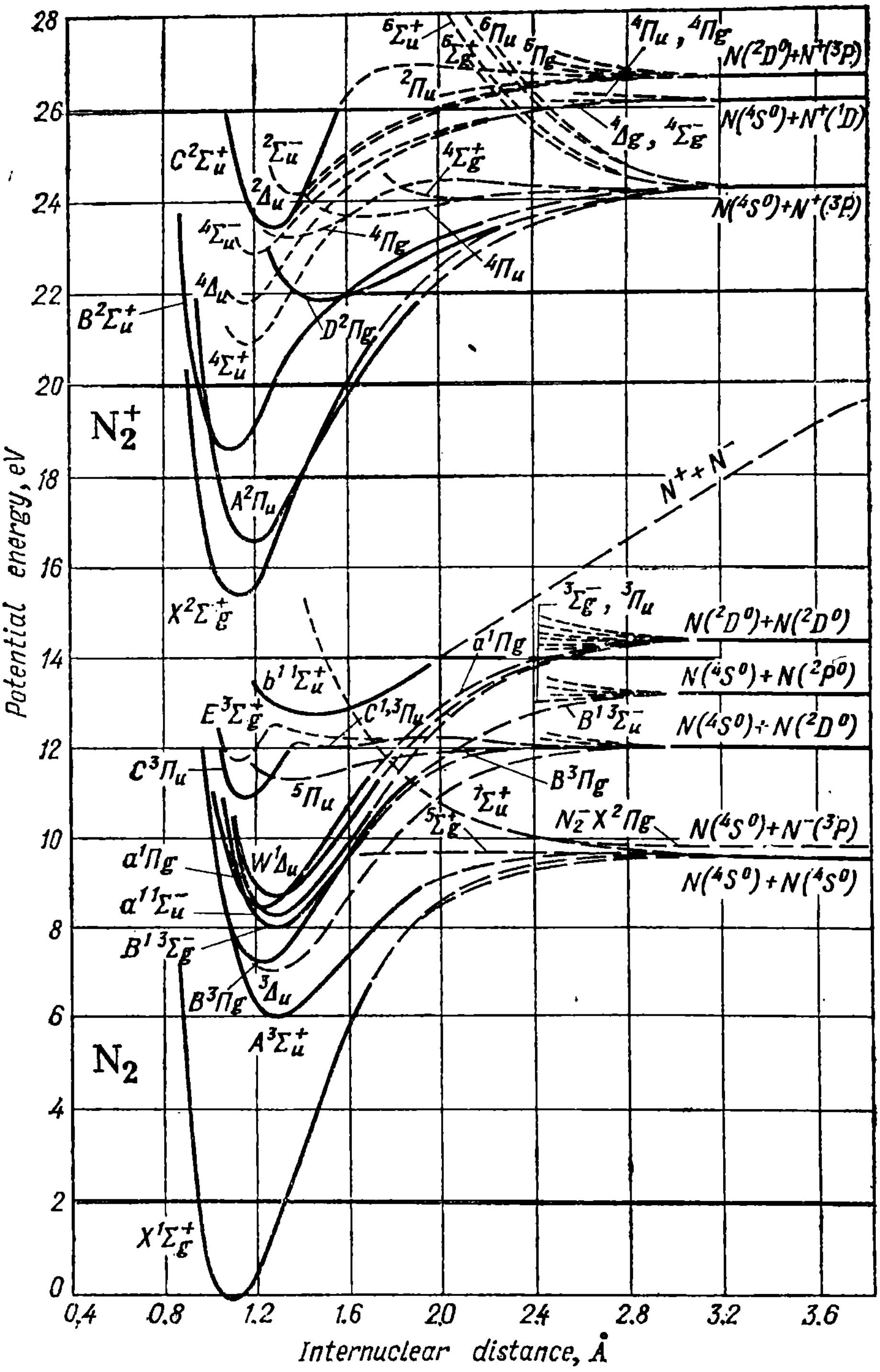

Fig. A13.1. Potential curves for the nitrogen molecule.

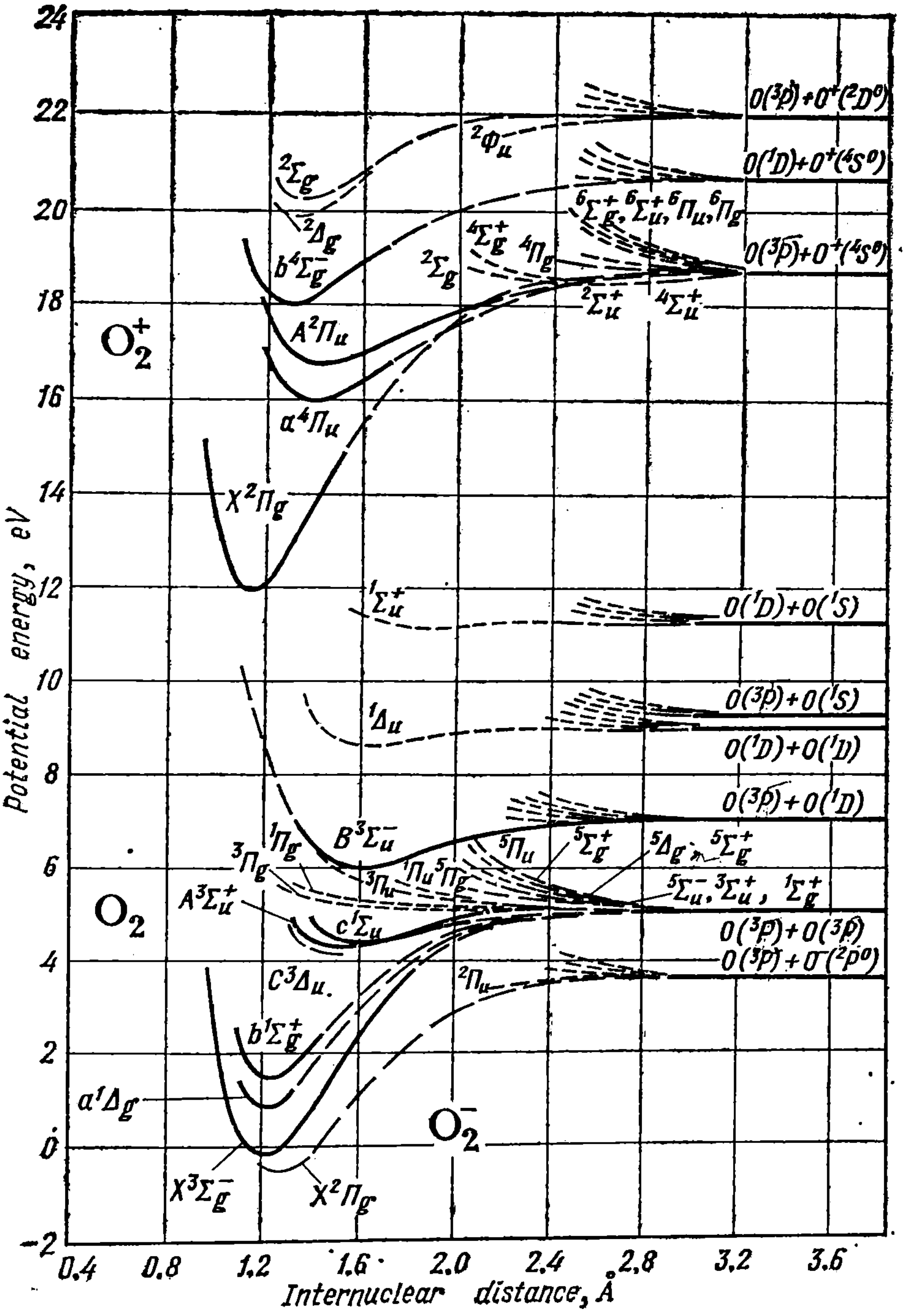

Fig. A13.2. Potential curves for the oxygen molecule.

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